THREE PROBLEMS IN ERGODIC THEORY.

by

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CONTENTS.

The three problems of the title of this thesis are dealt with in three sections. A page number of the form X - Y - Z refers to page Z of subsection Y of section X. A page in an undivided section is numbered X - Z.

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DECLARATION.

The work presented in section one is based, in part, on my M.Sc. dissertation and, in a slightly different form, was published as a joint paper with William Parry (6). It should be regarded as joint work to an extent greater than is implied by the relationship of supervisor and student.

Subsection one of section three is almost entirely expository. Only theorem 6 and the following discussion are original.

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SUMMARY.

The three problems refered to in the title of this thesis are investigated in three sections, which are entirely independent of each other. Ergodic Theory includes, in our view, Topological Dynamics and, in fact, section two is entirely topological and section three mostly so.

Section 1. The concept of a pair of σ-algebras being regular is introduced and, hence, a notion of isomorphism more restrictive than the usual conjugacy of measure preserving automorphisms is defined. This equivalence relation may be interpreted on the endomorphism level (theorem 1). In subsection 2 an invariant of the relation which is often finer than entropy is introduced.

Section 2. Following some work of Gurevic (8) a few simple facts about the topological entropy of sub-shifts of finite type on countably many symbols are derived. This enables us to give an example of a homeomorphism of a zero dimensional space which has both finite and infinite topological entropy, with respect to equivalent metrics.

Section 3. This section is divided into five subsections.

1. Following an account of Hahn (12) and Parry's (20) theory of topological group actions with quasi-discrete spectrum we show any transformation to which such an action is transversal is affine (theorem 6).
2. This subsection motivates the next three.
3. A fairly general method of constructing discrete actions of finitely generated abelian groups as affine transformations of finite dimensional tori is given. This method is designed to meet the needs of the proof of Weyl's theorem in subsection 5.

4. Under a mild hypothesis the actions constructed in (3) are shown to be totally ergodic, with respect to Haar measure, (theorem 8) and have quasi-discrete spectrum (theorem 9). We are therefore, in particular, able to give a general theorem (no. 10) about the existence of $\mathbb{Z}^m$ - actions to which the theory outlined in (1) applies.

5. The results of (3) and (4) are used to give a new proof of Weyl's theorem (28) on the uniform distribution of polynomials of integer variables.

<u>Numbering of Results.</u>

Theorems and propositions are numbered consecutively within each section. Lemmas are numbered consecutively within each subsection. When it is necessary to refer to a lemma in a previous subsection, say Y, then it is denoted lemma Y.Z, where Z is the number of the lemma.

Section One.

ON THE REGULARITY OF σ-ALGEBRAS AND CONJUGACIES.

1. Regularity.

In this section we consider endomorphisms (i.e. measure preserving transformations) and automorphisms (i.e. invertible endomorphisms) of Lebesgue spaces. We use a number of well known facts about Lebesgue spaces which are described in, for instance, (25). We often write equalities which are, in reality, equalities except on a set of measure zero. Let $\bar{T} : (\bar{X}, \bar{\mathcal{B}}, \bar{m}) \longrightarrow (\bar{X}, \bar{\mathcal{B}}, \bar{m})$ be an endomorphism of a Lebesgue space and $\mathcal{A} \subset \bar{\mathcal{B}}$ be a σ-algebra of measurable sets. If $\bar{T}^{-1}\mathcal{A} \subset \mathcal{A}$ one can construct another endomorphism, T, of a Lebesgue space, $(X, \mathcal{B}, m)$ which is known as the factor corresponding to $\mathcal{A}$. Here X is the measurable partition of $\bar{X}$ corresponding to $\mathcal{A}$, $\mathcal{B} = \mathcal{A}$ and $m(A) = \bar{m}(A)$ for all $A \in \mathcal{A}$. Then there is a commutative diagram;

$$\begin{tikzcd} \bar{X} \arrow[r, "\bar{T}"] \arrow[d, "\pi"'] & \bar{X} \arrow[d, "\pi"] \\ X \arrow[r, "T"] & X \end{tikzcd}$$

where π is the map which sends an element of $\bar{X}$ to the unique element of X containing it. Conversely given a measure preserving map π and endomorphisms satisfying such a commutative diagram T is conjugate to the factor of $\bar{T}$ corresponding to the σ-algebra $\pi^{-1}\mathcal{B}$. We therefore say that in this situation also T is the factor of $\bar{T}$ corresponding to $\pi^{-1}\mathcal{B}$.

We list next some standard notation which we will use.

<u>Notation.</u> If ξ is a measurable partition of $(\bar{X}, \bar{\mathcal{B}}, \bar{m})$ then $\hat{\xi}$ is the sub σ-algebra of $\bar{\mathcal{B}}$ consisting of unions of elements of ξ.

$\xi_1 \leq \xi_2$ if and only if $\hat{\xi}_1 \subset \hat{\xi}_2$.

$\bigvee_i \{\xi_i : i \in \mathbb{N}\}$ denotes the smallest measurable partition greater than all the ξ_i.

$\xi_i \nearrow \xi$ means $\xi = \bigvee_i \{\xi_i : i \in \mathbb{N}\}$ and for all i; $\xi_i \leq \xi_{i+1}$.

Regular isomorphism is an equivalence relation among endomorphisms of a Lebesgue space more restrictive than the usual idea of isomorphism (or conjugacy). The motivation for considering this equivalence relation is derived from some comments about coding.

In coding theory one considers a shift automorphism $\overline{T} : (\overline{X}, \overline{\mathcal{B}}, \overline{m}) \longrightarrow (\overline{X}, \overline{\mathcal{B}}, \overline{m})$. Here $(\overline{X}, \overline{\mathcal{B}}) = \prod_{-\infty}^{\infty} (A, \mathcal{P}(A))$, where A is a finite set (or alphabet) and $\overline{m}$ is a $\overline{T}$ - invarient probability measure on $(\overline{X}, \overline{\mathcal{B}})$. If we denote an element of $\overline{X}$ by $(\ldots x_{-1}, x_0, x_1, \ldots)$ then $(\overline{T}x)_i = x_{i+1}$. An element of $\overline{X}$ can be thought of as a message and $\overline{T}$ as the passing of one unit of time, in which one symbol of the message can be received or transmitted. In practice we would hope that each message is of finite, if unbounded, length as human life is similarly constrained. (We shall not, in fact, impose this restriction in our formal definition.)

A coding of messages received is their translation into bi-sequences each of whose entries is drawn from another finite alphabet A'. If we wish to be able to recover the original message we are led to consider an invertible measure preserving transformation $\phi : (\overline{X}, \overline{\mathcal{B}}, \overline{m}) \rightarrow (\overline{X}', \overline{\mathcal{B}}', \overline{m}')$ such that $\phi \overline{T} = \overline{T}' \phi$. Here $(\overline{X}', \overline{\mathcal{B}}') = \prod_{-\infty}^{\infty} (A', \mathcal{P}(A'))$, $\overline{m}'$ is a $\overline{T}'$ - invarient measure and $\overline{T}' : (\overline{X}', \overline{\mathcal{B}}', \overline{m}') \longrightarrow (\overline{X}', \overline{\mathcal{B}}', \overline{m}')$ is the shift automorphism.

It is not reasonable to expect a symbol of the encoded message to depend on all those of the original as this would mean waiting an infinite amount of time befor starting to code. If we assume that human (or machine) patience is finite, in fact bounded by N units of time, then we should consider only conjugacies, ϕ , such that there exists an integer $N \geq 0$ and $(\phi x)_0$ depends only on $\{x_i : -\infty \leq i \leq N\}$. Let $\mathcal{A}$, $\mathcal{A}'$ denote the sub σ-algebras of $\overline{\mathcal{B}}$, $\overline{\mathcal{B}}'$;

$$\mathcal{A} = \prod_{-\infty}^{0} \mathcal{P}(A) \qquad \mathcal{A}' = \prod_{-\infty}^{0} \mathcal{P}(A').$$

Then this extra condition on ϕ may be written $\overline{T}^{-N}\mathcal{A} \supset \phi^{-1}\mathcal{A}'$. We naturally assume this restriction is also placed on the decoding process; ϕ^{-1}. That is to say; $\overline{T}'^{-1}\mathcal{A}' \supset \phi\mathcal{A}$.

If, in fact, there is no delay at all in the encoding and decoding processes, so $N = 0$, then we are just considering the well known problem of conjugacy of the one sided shift endomorphisms; $T : (X, \mathcal{B}, m) \longrightarrow (X, \mathcal{B}, m)$ and $T' : (X', \mathcal{B}', m') \longrightarrow (X', \mathcal{B}', m')$. Here T is the factor of $\overline{T}^{-1}$ corresponding to the σ-algebra $\mathcal{A}$ and T' is defined similarly.

In the following definition the automorphism $\overline{T}$ may be thought of (in terms of motivation) as the inverse shift automorphism.

<u>Definition.</u> Let $\overline{T} : (\overline{X}, \overline{\mathcal{B}}, \overline{m}) \to (\overline{X}, \overline{\mathcal{B}}, \overline{m})$ be an automorphism of a Lebesgue space. A pair of sub σ-algebras of $\overline{\mathcal{B}}$; $\mathcal{A}_1$ and $\mathcal{A}_2$ are <u>regular</u> if;

(1) $\overline{T}^{-1}\mathcal{A}_i \subset \mathcal{A}_i \qquad i = 1,2$

(2) $\overline{T}^{n}\mathcal{A}_i \nearrow \mathcal{B} \qquad i = 1,2$

(3) $\exists$ an integer $N \geq 0$ such that $\overline{T}^{N}\mathcal{A}_i \supset \mathcal{A}_j$, $\{i,j\} = \{1,2\}$.

<u>Definition.</u> Let $\overline{T}_i : (\overline{X}_i, \overline{\mathcal{B}}_i, \overline{m}_i) \to (\overline{X}_i, \overline{\mathcal{B}}_i, \overline{m}_i)$, $i = 1,2$, be automorphisms of Lebesgue spaces and $\mathcal{A}_i \subset \overline{\mathcal{B}}_i$ sub σ-algebras. Then an invertible measure preserving transformation $\phi : \overline{X}_1 \to \overline{X}_2$ is <u>regular</u>, with respect to $\mathcal{A}_1$ and $\mathcal{A}_2$, if;

(1) $\phi \overline{T}_1 = \overline{T}_2 \phi$.

(2) $\mathcal{A}_1$ and $\phi^{-1}\mathcal{A}_2$ are a regular pair of σ-algebras.

If the sub σ-algebras under consideration are clear then we say $\overline{T}_1$ and $\overline{T}_2$ are regularly isomorphic.

<u>Definition.</u> Let $T_i : (X_i, \mathcal{B}_i, m_i) \longrightarrow (X_i, \mathcal{B}_i, m_i)$, $i = 1,2$, be endomorphisms of Lebesgue spaces. They are <u>shift equivalent</u> with lag K if there exists an integer $K \geqslant 0$ and measure preserving maps $\Theta : X_1 \to X_2$ and $\Upsilon : X_2 \to X_1$ such that;

(1) $\Theta T_1 = T_2 \Theta$ and $\Upsilon T_2 = T_1 \Upsilon$.

(2) $\Upsilon \Theta = T_1^K$ and $\Theta \Upsilon = T_2^K$.

This last concept is due to R.F. Williams (29) and enables us to interpret regular isomorphism on the level of endomorphisms. Specifically the next result generalises the remark that if $N = 0$ then one is dealing with conjugacy of one-sided shifts.

An endomorphism T has natural extension $\overline{T}$ if T is a factor;

$$\begin{tikzcd}
(\overline{X}, \overline{\mathcal{B}}, \overline{m}) \arrow[r, "\overline{T}"] \arrow[d, "\pi"'] & (\overline{X}, \overline{\mathcal{B}}, \overline{m}) \arrow[d, "\pi"] \\
(X, \mathcal{B}, m) \arrow[r, "T"] & (X, \mathcal{B}, m)
\end{tikzcd}$$

such that $\overline{T}$ is an automorphism and $\overline{T}^n(\pi^{-1}\mathcal{B}) \nearrow \overline{\mathcal{B}}$. Rohlin

(24) has shown that every endomorphism has a natural extension which is unique up to conjugacy. Furthermore the entropies $h(T)$ and $h(\overline{T})$ are equal. Conjugate endomorphisms have, of course, conjugate natural extensions though a single automorphism can be the natural extension of many non-isomorphic endomorphisms.

<u>Theorem 1.</u> Let $\overline{T} : (\overline{X}, \overline{\mathcal{B}}, \overline{m}) \longrightarrow (\overline{X}, \overline{\mathcal{B}}, \overline{m})$ be an automorphism of a Lebesgue space and $\mathcal{A}_1, \mathcal{A}_2 \subset \overline{\mathcal{B}}$ be a regular pair of sub σ-algebras. Then the factor endomorphisms, T_i, corresponding to $\mathcal{A}_i$, $i = 1,2$, are shift equivalent.

Conversely shift equivalent endomorphisms $T_i : (X_i, \mathcal{B}_i, m_i) \longrightarrow (X_i, \mathcal{B}_i, m_i)$ have the same natural extension $\overline{T} : (\overline{X}, \overline{\mathcal{B}}, \overline{m}) \longrightarrow (\overline{X}, \overline{\mathcal{B}}, \overline{m})$ and correspond to a pair of regular sub σ-algebras of $\overline{\mathcal{B}}$.

<u>Proof</u>. If ξ_1 and ξ_2 are measurable partitions and $\xi_1 \leq \xi_2$ then one can define a map $a : \xi_2 \longrightarrow \xi_1$, where $a(A) \supset A$.

We shall always denote such a map $\xi_2 \hookrightarrow \xi_1$.

For instance if we regard T_i as endomorphisms of the space ξ_i, where $\hat{\xi}_i = \mathcal{A}_i$, then we have;

$$\begin{tikzcd} \xi_i \arrow[rr, "\overline{T}"] \arrow[dr, "T_i"'] & & \overline{T}(\xi_i) \arrow[dl, hook] \\ & \xi_i & \end{tikzcd}$$

Since $\overline{T}^N(\xi_i) \geqslant \xi_j$ we can define measure preserving maps $\theta : X_1 \longrightarrow X_2$ and $\Psi : X_2 \longrightarrow X_1$ by the following diagrams;

$$\begin{tikzcd} \xi_1 \arrow[rr, "\bar{T}^N"] \arrow[dr, "\theta"'] & & \bar{T}^N(\xi_1) \arrow[dl, hook] \\ & \xi_2 & \end{tikzcd}$$

$$\begin{tikzcd} \xi_2 \arrow[rr, "\bar{T}^N"] \arrow[dr, "\Upsilon"'] & & \bar{T}^N(\xi_2) \arrow[dl, hook] \\ & \xi_1 & \end{tikzcd}$$

Then $\Upsilon \circ \theta$ is the mapping;

$$\xi_1 \xrightarrow{\bar{T}^N} \bar{T}^N(\xi_1) \hookrightarrow \xi_2 \xrightarrow{\bar{T}^N} \bar{T}^N(\xi_2) \hookrightarrow \xi_1$$

Or;

$$\xi_1 \xrightarrow{\bar{T}^{2N}} \bar{T}^{2N}(\xi_1) \hookrightarrow \xi_1 .$$

I.e. $\Upsilon \circ \theta = T_1^{2N}$.

Similarly $\theta \circ \Upsilon = T_2^{2N}$ and clearly $\theta T_1 = T_2 \theta$ and $\Upsilon T_2 = T_1 \Upsilon$ so we have shown the factors are shift equivalent with lag $2N$.

Conversely let T_1 be shift equivalent with lag K, as in the definition.

Let $\bar{T} : (\bar{X}, \bar{\mathcal{B}}, \bar{m}) \to (\bar{X}, \bar{\mathcal{B}}, \bar{m})$ be the natural extension of T_1. Then there exists a measure preserving map $\pi : \bar{X} \to X_1$ such that $\bar{T}^n(\pi^{-1} \mathcal{B}_1) \nearrow \bar{\mathcal{B}}$ and the following diagram commutes;

$$\begin{tikzcd}
\overline{X} \arrow[r, "\overline{T}"] \arrow[d, "\overline{\pi}"'] & \overline{X} \arrow[d, "\overline{\pi}"] \\
X_1 \arrow[r, "T_1"] \arrow[d, "\theta"'] & X_1 \arrow[d, "\theta"] \\
X_2 \arrow[r, "T_2"] & X_2
\end{tikzcd}$$

T_2 is the factor of $\overline{T}$ corresponding to the σ-algebra $\pi^{-1}\theta^{-1}(\mathcal{B}_2)$.

Now $\pi^{-1}\theta^{-1}(\mathcal{B}_2) \supset \pi^{-1}\theta^{-1}\gamma^{-1}(\mathcal{B}_1) = \overline{T}^{-K}(\pi^{-1}\mathcal{B}_1)$.

So $\overline{T}^{n}(\pi^{-1}\theta^{-1}(\mathcal{B}_2)) \nearrow \overline{\mathcal{B}}$ and $\overline{T}$ is also the natural extension of T_2.

Finally notice that $\pi^{-1}\mathcal{B}_1 \supset \pi^{-1}\theta^{-1}\gamma^{-1}\theta^{-1}(\mathcal{B}_2) = \overline{T}^{-K}(\pi^{-1}\theta^{-1}(\mathcal{B}_2))$, completing the proof.

2. An Invariant.

In this subsection we introduce an invarient of regularity. It can, of course, be thought of a an invariant of the relation of regular isomorphism. Conjugate automorphisms have, of course, the same entropy and the numerical invariant we describe below is often finer than that of having equal entropy.

We refer the reader to (19) and (23) for a much fuller discussion of entropy theory but include here a short summary of the results used.

If ξ is a countable measurable partition of a Lebesgue space $(X, \mathcal{B}, m)$ and $\mathcal{C} \subset \mathcal{B}$ we define the information function;

$$I(\xi / \mathcal{C}) = - \sum_{A \in \xi} \chi_A \log m(A / \mathcal{C})$$

where $m(\ /\mathcal{C})$ is the conditional measure. One sometimes writes $\hat{\xi}$ instead of ξ. Let;

$$H(\xi / \mathcal{C}) = \int_X I(\xi / \mathcal{C})\, dm$$

and $H(\xi)$ denote $H(\xi / \mathcal{C})$ when $\mathcal{C}$ is the trivial σ-algebra having only sets of measure one and zero. We define the entropy of an endomorphism, T, to be;

$$h(T) = \sup \lim_{n \to \infty} 1/n\, H\Big(\bigvee_0^n T^{-i} \xi\Big),$$

where the supremum is over finite measurable ξ. If $h(T)$ is finite and $\mathcal{C} \subset \mathcal{B}$ has the property $T^{-1}\mathcal{C} \subset \mathcal{C}$ then there is a countable measurable partition ξ such that $T^{-1}\mathcal{C} \vee \hat{\xi} = \mathcal{C}$. In this case we let $I(\mathcal{C} / T^{-1}\mathcal{C}) = I(\xi / T^{-1}\mathcal{C})$, which is well defined. We have;

$$h(T) \geqslant H(\mathcal{C} / T^{-1}\mathcal{C}) = \int_X I(\mathcal{C} / T^{-1}\mathcal{C})\, dm.$$

Finally we note three simple facts;

(1) $I(\xi_1 \vee \xi_2 / \mathcal{C}) = I(\xi_2 / \mathcal{C}) + I(\xi_1 / \hat{\xi}_2 \vee \mathcal{C})$

(2) $I(\xi / \mathcal{C}) \geqslant 0$

(3) $I(T^{-1}\xi / T^{-1}\mathcal{C}) = I(\xi / \mathcal{C})\circ T$

Definition. A coboundary with respect to an endomorphism, $T : X \to X$ is a function of the form $h\circ T - h$, where $h : X \to \mathbb{C}$ is measurable. If the T-invarient measure under consideration is m and $h \in L^2(m)$ then $h\circ T - h$ is a L^2 - coboundary. Two measurable complex valued functions, f and g, are cohomologous (resp. L^2 - cohomologous) if $f - g$ is a coboundary (resp. $f, g \in L^2(m)$ and $f - g$ is a L^2 - coboundary).

Since the process of passing to a natural extension preserves entropy we know, by theorem 1, that entropy is an invarient of shift equivalence. The next result is often a strengthening of this remark.

Theorem 2. Let $\bar{T} : (\bar{X}, \bar{\mathcal{B}}, \bar{m}) \to (\bar{X}, \bar{\mathcal{B}}, \bar{m})$ be an automorphism of a Lebesgue space and $\mathcal{A}_1, \mathcal{A}_2$ be a regular pair of σ-algebras. If $h(\bar{T})$ is finite then $I(\mathcal{A}_1 / \bar{T}^{-1}\mathcal{A}_1)$ and $I(\mathcal{A}_2 / \bar{T}^{-1}\mathcal{A}_2)$ are cohomologous.

<u>Proof.</u> $I(\mathcal{A}_1 / \bar{T}^{-N}\mathcal{A}_2) \leq I(\bar{T}^{N}\mathcal{A}_2 / \bar{T}^{-N}\mathcal{A}_2)$ by fact (1)

$$= I(\bar{T}^{N}\mathcal{A}_2 \vee \bar{T}^{N-1}\mathcal{A}_2 \vee \dots \vee \bar{T}^{-(N-1)}\mathcal{A}_2 / \bar{T}^{-N}\mathcal{A}_2)$$

$$= \sum_{j=-N}^{N-1} I(\mathcal{A}_2 / \bar{T}^{-1}\mathcal{A}_2)\circ T^{j} \qquad \text{by (1) and (3)}$$

$$< \infty .$$

Thus we can calculate;

$$I(\mathcal{A}_1 / \bar{T}^{-(N+1)}\mathcal{A}_2) = I(\mathcal{A}_1 \vee \bar{T}^{-N}\mathcal{A}_2 / \bar{T}^{-(N+1)}\mathcal{A}_2)$$

$$= I(\bar{T}^{-N}\mathcal{A}_2 / \bar{T}^{-(N+1)}\mathcal{A}_2) + I(\mathcal{A}_1 / \bar{T}^{-N}\mathcal{A}_2) \quad \text{by (1).}$$

and also $I(\mathcal{A}_1 / \bar{T}^{-(N+1)}\mathcal{A}_2) = I(\mathcal{A}_1 \vee \bar{T}^{-1}\mathcal{A}_1 / \bar{T}^{-(N+1)}\mathcal{A}_2)$

$$= I(\bar{T}^{-1}\mathcal{A}_1 / \bar{T}^{-(N+1)}\mathcal{A}_2) + I(\mathcal{A}_1 / \bar{T}^{-1}\mathcal{A}_1) \quad \text{by (1).}$$

In other words;

$$I(\mathcal{A}_2 / \bar{T}^{-1}\mathcal{A}_2)\circ\bar{T}^{N} + I(\mathcal{A}_1 / \bar{T}^{-N}\mathcal{A}_2) =$$

$$I(\mathcal{A}_1 / \bar{T}^{-N}\mathcal{A}_2)\circ\bar{T} + I(\mathcal{A}_1 / \bar{T}^{-1}\mathcal{A}_1) \qquad \text{by (3).}$$

We are in a situation where $f\circ\bar{T}^{N} - g = h\circ\bar{T} - h$.

Then;

$$f - g = h\circ\bar{T} - h + (f - f\circ\bar{T}) + (f\circ\bar{T} - f\circ\bar{T}^{2}) + \dots$$
$$\dots + (f\circ\bar{T}^{N-1} - f\circ\bar{T}^{N})$$

and the right hand side is certainly a coboundary.

<u>Corollary.</u> The first part of the proof shows that if $I(\mathcal{A}_1 / \bar{T}^{-1}\mathcal{A}_1)$ and $I(\mathcal{A}_2 / \bar{T}^{-1}\mathcal{A}_2)$ are elements of $L^2(m)$ then they are L^2-cohomologous.

Let $I_i = I(\alpha_i / \bar{T}^{-1}\alpha_i) - \int_X I(\alpha_i / \bar{T}^{-1}\alpha_i)\, dm.$

Also define the following real number if the limit exists;

$$V_i = \int_X I_i^2\, dm + 2 \lim_{n\to\infty} 1/n \sum_{j=1}^{n} (n-j) \int_X (I_i \circ T^j).I_i\, dm$$

We claim that, under the conditions of the last corollary, this is a numerical invariant of regularity. In fact we prove the following result which can be applied to the space of real valued square integrable functions.

<u>Lemma 1.</u> Let U be an isometry of a real Hilbert space and $x,y,z \in H$ satisfy the equation $x - y = Uz - z$.
Then;

$$V(x) = \langle x,x\rangle + 2 \lim_{n\to\infty} 1/n \sum_{j=1}^{n} (n-j) \langle U^j x, x\rangle$$

exists if and only if $V(y)$ exists, and in this case they are equal.

<u>Proof.</u> Let $S_n = x + Ux + \dots + U^{n-1}x$ and

$T_n = y + Uy + \dots + U^{n-1}y.$

Then $S_n - T_n = U^n z - z$ so;

$$\langle S_n, S_n\rangle - \langle T_n, S_n\rangle = \langle U^n z - z, S_n\rangle$$

$$\langle S_n, T_n\rangle - \langle T_n, T_n\rangle = \langle U^n z - z, T_n\rangle$$

Adding these two equations;

$$\| S_n\|^2 - \| T_n\|^2 = \langle U^n z - z, S_n + T_n\rangle$$

I.e. $n.V_n(x) - n.V_n(y) = \langle U^n z - z, S_n + T_n\rangle,$

where $V_n(x) = \langle x,x\rangle + 2/n \sum_{j=1}^{n} (n-j)\langle U^j x, x\rangle.$

To prove the lemma it suffices to show that;

$$1/n\langle U^n z - z\ ,\ S_n\rangle \to 0 \qquad 1/n\langle U^n z - z\ ,\ T_n\rangle \to 0$$

as n tends to infinity.

We prove the first of these two since the argument is the same in the second case.

From the mean ergodic theorem (see (13)) we know that S_n/n converges to the projection of x onto the subspace of vectors fixed by U.

Now;

$$\begin{aligned}\langle U^n z - z\ ,\ S_n\rangle &= \langle U^n z, S_n\rangle - \langle z, S_n\rangle \\ &= \langle U^n z\ ,\ S_n - U^n S_n\rangle \\ &= \langle U^n z\ ,\ 2.S_n - S_{2n}\rangle\end{aligned}$$

Therefore;

$$|\,1/n\langle U^n z - z\ ,\ S_n\rangle\,| \leqslant \|z\|\,\|(1/n)2.S_n - (1/2n)2.S_{2n}\|$$

and the right hand side tends to zero.

Using both this invariant (6) and another (22) Parry has given [redacted] examples of conjugacies which are not regular.
For instance let T_1 and T_2 be the Meshalkin examples of Bernoulli shifts with weights $(1/4, 1/4, 1/4, 1/4)$ and $(1/2, 1/8, 1/8, 1/8, 1/8)$. Then it is easy to see $V_1 = 0$ but $V_2 \neq 0$.

Section Two.

ON THE TOPOLOGICAL ENTROPY OF DENUMERABLE SUB-SHIFTS OF FINITE TYPE.

We adopt Bowen's approach to topological entropy (3) as described in (27) where full details may be found. If $T: X \to X$ is a uniformly continuous homeomorphism of a metric space, $K \subset X$ is compact, $n \in \mathbb{N}$ and $\varepsilon > 0$ we say $F \subset K$ is (n,ε) spanning for K if for all $x \in K$ there exists $y \in F$ such that;

$$\max_{0 \leq i \leq n-1} d(T^i x, T^i y) < \varepsilon \quad .$$

Let $r_n(\varepsilon,K)$ denote the minimum cardinality of such a set F and define the topological entropy of T with respect to the metric d to be;

$$h_d(\ ) \quad = \quad \sup_{K \text{ compact}} \ \limsup_{n \to \infty} 1/n \log r_n(\varepsilon,K)$$

This number depends on the uniform equivalence class of the metric. In fact, as is well known, it is easy to construct a homeomorphism of an infinite dimensional space, $\mathbb{R}^{\infty}$, whose entropy with respect to two metrics which define the same topology is zero in one case and infinite in the other. One considers the homeomorphism $f: \mathbb{R} \to \mathbb{R}$, $f(x) = 2x$, and metrics d and d' on the real line, where d is the usual euclidean distance and d' the metric inherited from regarding $\mathbb{R} \subset \mathbb{K}$ as a subset of its one point compactification. $h_d(f) = \log 2$, $h_{d'}(f) = 0$ and the homeomorphism T is the countably infinite direct product of f's.

The original definition of topological entropy, due to Adler, Konheim and McAndrew (2), only applies to continuous transformations of compact spaces. The two definitions are equivalent if X is both compact and metrisable. One advantage of Bowen's approach is, then, that it extends the concept of topological entropy to non-compact metric spaces. This had been done previously by Gurevic in the special case of sub-shifts of finite type (also known as intrinsic markov chains or topological markov chains) on countably many symbols (8). We note below (Proposition 1) that this is a special case of Bowen's extension.

Let $H = (h_{ij})$, $i,j \in \mathbb{N}$, be a countable matrix of zeroes and ones and define;

$$X(H) = \left\{ x = (\dots x_{-1}, x_0, x_1, \dots) : \forall i \in \mathbb{Z}\ h_{x_i x_{i+1}} = 1 \right\},$$

a subset of $\prod_{-\infty}^{\infty} \mathbb{N}$. Then the shift map $T_H : X(H) \to X(H)$, where $(T_H(x))_i = x_{i+1}$ is of the type considered by Gurevic. We define a cylinder set;

$$(j_0, \dots j_n) = \left\{ x \in X : x_i = j_i,\ 0 \leq i \leq n \right\}.$$

and assume that T_H is transitive in the sense; for each $i,j \in \mathbb{N}$ there exist cylinder sets $(i, j_1, \dots j_{n-1}, j)$ and $(j, j'_1, \dots j'_{n'-1}, i)$. If $N \subset \mathbb{N}$ one may define a sub-matrix H_N of H; $H_N = (h_{ij})$, $i,j \in N$ and hence the space $X_N \subset X(H)$ and $T_N : X_N \to X_N$. If N is a finite set and T_N is transitive then these are the sub-shifts of finite type studied by Parry (21).

We use Gurevic's work to show that in the zero dimensional case also the property of having finite topological entropy is not necessarily preserved in passing from one metric to an equivalent one. Specifically we define two equivalent metrics d and d' and a sub-shift of finite type T_H such that $h_{d'}(T_H) = \log 2$ and $h_d(T_H) = \infty$.

Consider the following topologies on $X(H)$:

T_1: The topology as a subset of $\prod_{-\infty}^{\infty} \mathbb{N}$, with the product topology, where $\mathbb{N}$ has the discrete topology.

T_2: The topology derived from the metric d;

$$d(x,y) = \sum_{i=-\infty}^{\infty} (1-\delta(x_i,y_i))/ 2^{|i|},$$

where $\delta(x_i,y_i) = 1$ if $x_i = y_i$ and is zero if not.

T_3: The topology derived from the metric d';

$$d'(x,y) = \sum_{i=-\infty}^{\infty} (1-\delta(x_i,y_i))/ 2^{|i|}.\min(x_i,y_i) .$$

<u>Lemma 1.</u> $T_1 = T_2 = T_3$.

<u>Proof.</u> We use the fact that a base $\mathcal{U}$ defines a weaker topology than a base $\mathcal{V}$ if for all $U \in \mathcal{U}$ and $u \in U$ there exists $V \in \mathcal{V}$ such that $u \in V \subset U$.

Let $A = \{ y : y_{i_n} = j_n \}$ for some $i_1, \dots i_m \in \mathbb{Z}$ and $j_1, \dots j_m \in \mathbb{N}$ be a typical basic open set in T_1.

Let $J = \max(j_n)$, $L = \max(|i_n|)$ and choose $0 < \varepsilon < 1/ 2^L.J$.

Then for any $x \in A$; $A \supset \{ y : d'(x,y) < \varepsilon \} \in T_3$.

Since $d(x,y) \geqslant d'(x,y)$ for all $x,y \in X(H)$ T_3 is weaker than T_2.

Let $B = \{ y : d(x,y) < \varepsilon \}$ be an open ball in T_2 and choose L so that $\sum 1/ 2^{|i|} < \varepsilon$, where the sum is over $|i| > L$.

If $x \in B$ define $A = \{ y : y_i = x_i, -L \leq i \leq L \}$.

Then $x \in A \subset B$.

It is clear that $d(x,y) \le \varepsilon$ implies $d(T_H x, T_H y) \le 2\varepsilon$ and similarly $d'(x,y) \le \varepsilon$ implies $d'(T_H x, T_H y) \le 2\varepsilon$ so T_H is uniformly continuous with respect to both metrics. Since any cylinder set is both open and closed we see the topology makes $X(H)$ zero dimensional. One may compactify $\mathbb{N}$ by adjoining the point ∞. Let $\overline{\mathbb{N}}$ denote this space and $\overline{X(H)}$ the closure of $X(H)$ in $\prod_{-\infty}^{\infty} \overline{\mathbb{N}}$. The topology on $\prod_{-\infty}^{\infty} \overline{\mathbb{N}}$ is given by the metric d', where $\min(j,\infty) = j$, $\min(\infty,\infty) = \infty$ and $1/\infty = 0/\infty = 0$. One has $T_H : \overline{X(H)} \to \overline{X(H)}$, a homeomorphism of a compact metric space. If $h_{top}(T_H)$ denotes the topological entropy of this homeomorphism then the main result of (8) is the following one.

<u>Theorem</u> (B.M. Gurevic) $h_{top}(T_H) = \sup h(T_N)$, where the supremum is over finite $N \subset \mathbb{N}$. Furthermore one may take the supremum over N which are finite and define transitive sub-shifts of finite type.

Gurevic defines the topological entropy of $T_H : X(H) \to X(H)$ to be number defined by the theorem. Since our first aim is to show that Bowen's definition is equivalent we characterise compact subsets of $X(H)$.

<u>Lemma 2.</u> A set $K \subset X(H)$ is compact if and only if the subset of $\mathbb{N}$ consisting of i'th co-ordinates of elements of K is finite, for all $i \in \mathbb{Z}$, and K is closed.

<u>Proof.</u> If K is compact then $\{(j) : j \in \mathbb{N}\}$ is an open cover of K and so has a finite sub-cover.

I.e. only a finite number of elements of $\mathbb{N}$ are 0'th co-ordinates of elements of K.

Obviously this argument works if $i \not< 0$ too.

Conversely let K have the properties indicated.

Let $Y = \{ y \in X(H) : \forall i \ \exists x \in K$ such that $y_i = x_i \}$

and $J_i = \max(x_i : x \in K)$ for each i.

If $\overline{X(H)} \ni z \notin Y$ and, more precisely, $z_n \notin \{ x_n : x \in K \}$

then for any $y \in Y$; $d'(z,y) \geqslant 1/ 2^{|n|} J_n$.

Thus Y is a closed subset of $\overline{X(H)}$, so Y is compact.

Therefore Y is also a compact subset of $X(H)$ and so

K is also.

In particular, of course, any subset of $X(H)$ of the form $\{ x : x_i \in N_i \}$ where N_i is a finite set for all $i \in \mathbb{Z}$ is compact.

<u>Proposition 1.</u> $h_{d'}(T_H) = h_{top}(T_H)$.

<u>Proof.</u> For any finite $N \subset \mathbb{N}$ let $K_N = \{ x : x_i \in N \ \forall i \}$.

Then, in the d' metric;

$$\limsup_{n \to \infty} 1/n \log r_n(T_H, \varepsilon, K_N) = h_{d'}(T_N).$$

Hence $\sup\limits_{N \text{ finite}} h_{d'}(T_N) \leq h_{d'}(T_H)$.

If K is any compact subset of $X(H)$ then it is also a compact subset of $\overline{X(H)}$. Thus;

$$\limsup_{n \to \infty} 1/n \log r_n(T_H, \varepsilon, K) \leq h_{top}(T_H) .$$

This completes the proof, using Gurevic's theorem.

<u>Notation.</u> For each $n, j \in \mathbb{N}$ we denote by $n(j)$ the number of cylinder sets in $X(H)$ or X_N (depending on the context) of the form; $(j, j_1, \ldots j_n)$.

<u>Lemma 3.</u> Let T_N be a transitive sub-shift of finite type, where N is finite. Then the entropy is given by;

$$h(T_N) = \limsup_{n\to\infty} 1/n \log n(j) ,$$

for any $j \in N$.

<u>Proof.</u> Consider $j, i \in N$. There is a cylinder set of the form $(i, j_1, \ldots j_{m-1}, j)$. Thus $n(i) \geq (n-m)(j)$, so;

$$\limsup_{n\to\infty} 1/n \log n(i) \geq \limsup_{n\to\infty} 1/n \log (n-m)(j)$$

$$= \limsup_{n\to\infty} 1/n-m \log (n-m)(j).$$

Thus $\limsup_{n\to\infty} 1/n \log n(j)$ is independant of j.

It is shown in (21) that $h(T_N) = \limsup_{n\to\infty} 1/n \log(\sum_{j\in N} n(j))$

There is a subsequence $n_i \to \infty$ and an element j_0 of N such that $\forall j \in N$; $n_i(j_0) \geq n_i(j)$ and further;

$$h(T_N) = \lim_{n_i\to\infty} 1/n_i \log(\sum_{j\in N} n_i(j)) .$$

If the set N has p elements then;

$$h(T_N) \leq \limsup_{n_i\to\infty} 1/n_i \log(p.n_i(j_0))$$

$$= \limsup_{n_i\to\infty} 1/n_i \log n_i(j_0) \leq h(T_N) .$$

If N is not finite so X_N is not compact then we have to consider the possibility that the topological entropy is different when calculated with respect to different metrics. The next proposition can be regarded as a generalisation of lemma 3. to the countable case. In certain cases the first of the two inequalities may be deduced from Theorem 5 of a second paper of Gurevic (9).

Proposition 2. Let $T_H : X(H) \to X(H)$ be a transitive sub-shift of finite type. Then for any $j \in \mathbb{N}$;

$$h_{d'}(T_H) \leq \limsup_{n \to \infty} 1/n \log n(j) \leq h_d(T_H).$$

Proof. Just as is lemma 3 the choice of j is irrelevant.

Let N_k be a sequence of finite subsets of $\mathbb{N}$ such that each T_{N_k} is transitive and $h(T_{N_k}) \longrightarrow h_{d'}(T_H)$.

We may assume there exists $j \in \bigcap_k N_k$.

Let $n(j)$ and $n(j)_k$ refer to cylinder sets in $X(H)$ and X_{N_k} respectively.

For all k; $n(j) \geqslant n(j)_k$ so;

$$\limsup_{n \to \infty} 1/n \log n(j) \geqslant \limsup_{n \to \infty} 1/n \log n(j)_k .$$

This proves the first inequality.

We consider two possibilities to prove the second part.

First suppose that (j) is compact for all $j \in \mathbb{N}$.

I.e. for all j $n(j)$ is always finite.

Then if $\varepsilon < \frac{1}{2}$, with respect to the metric d we have;

$r_n(T_H, \varepsilon, (j)) \geq n(j)$ so $h_d(T_H) \geq \limsup_{n\to\infty} 1/n \log n(j)$.

Alternatively fix some $m, j \in \mathbb{N}$ with $m(j) = \infty$.

Then there exists a pair-wise disjoint collection of cylinder sets; $\left\{(z_{r0}, z_{r1}, \dots z_{rm}) : r \in \mathbb{N}^+, \; z_{r0} = j\right\}$.

There also exists a cylinder set $(y_0, y_1, \dots y_M)$, where $M \geq 1$ and $y_0 = y_M = j$.

For any integer P one may find a compact set $K_P \subset X(H)$ such that for all positive integers q, K_P contains every point x such that;

$$x_{kM+i} = y_i \qquad 0 \leq i \leq M-1, \text{ if } k < q$$

$$x_{qM+i} = z_{ri} \quad \text{for some } r \leq P^q, \; 0 \leq i \leq m.$$

Then if $n(j)_P$ refers to cylinder sets in K_P we have $(qM + m)(j) \geq P^q$.

Then if $\epsilon < \frac{1}{2}$, with respect to the metric d we have;

$$h_d(T_H) \geq \limsup_{n\to\infty} 1/n \log r_n(T_H, \varepsilon, K_P)$$

$$\geq \limsup_{n\to\infty} 1/n \log n(j)_P \geq 1/M \log P .$$

Therefore in this case $h_d(T_H) = \infty = \limsup_{n\to\infty} 1/n \log n(j)$.

We now turn to the definition of the particular sub-shift of finite type, T_H, promised at the bottom of page 2 - 2. Let $H = (h_{ij})$, $i, j \in \mathbb{N}$, where $h_{0j} = 1$ for all j, $h_{i\,i-1} = 1$, for all i and the other entries are all zero. Since $n(0)$ is infinite for all $n \geq 1$ we know immediately that $h_d(T_H)$ is infinite.

Let $N_n = \{0,1, \dots n-1\}$ so T_{N_n} is the transitive sub-shift of finite type defined by the matrix n by n matrix H_{N_n};

$$\begin{matrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \dots & 0 & 1 & 0 \end{matrix}$$

Since $\bigcup_n N_n = \mathbb{N}$, $h_{d'}(T_N) = \lim_{n\to\infty} h(T_{N_n})$. The fact that this limit is $\log 2$ follows from the next two lemmas and the theorem of Parry (21) that the topological entropy of a transitive sub-shift of finite type on finitely many symbols is equal to $\log\lambda$, where λ is the eigenvalue of the defining matrix of greatest modulus. The existence of λ and the fact it is positive and real is proved in, for instance, (7).

Lemma 4. The characteristic polynomial of H_{N_n} is;

$$f_n(x) = x^n - x^{n-1} - x^{n-2} - x^{n-3} - \dots - x - 1 .$$

Proof. Let A_n be the n by n matrix having first row consisting of 1's, a sub-diagonal of 1's, the rest of the diagonal -x's and remaining entries 0. Thus;

$$\begin{matrix} 1 & 1 & 1 & 1 & 1 & \dots & 1 \\ 1 & -x & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & -x & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & -x & 0 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \dots & 0 & 0 & 0 & 1 & -x \end{matrix}$$

We claim $\det(A_n) = (-1)^{n-1}.(x^{n-1} + x^{n-2} + \ldots + x + 1)$.

Clearly $\det(A_2) = -(x + 1)$.

Assume, as an induction hypothesis, the claim is true for some $n \geqslant 2$.

Then $\det(A_{n+1}) = (-x)^n - \det(A_n)$

$= (-1)^n.(x^n + x^{n-1} + \ldots + x + 1)$

Now, if I_n denotes the n by n identity matrix, we have

$$\det(H_{N_n} - x.I_n) = (1-x).(-x)^{n-1} - \det(A_{n-1})$$

$$= (-1)^n.x^n + (-1)^{n-1}.(x^{n-1} + x^{n-2} + \ldots + 1)$$

$$= \pm f_n(x).$$

Lemma 5. Let x_0 be the largest real root of the equation $f_n(x) = 0$. Then; $2 - 1/2^{n-1} < x_0 < 2$, if n is large.

Proof. $f_n(x) = x^n + (1 - x^n)/(x - 1)$.

For all n; $f_n(2) = 1$.

Clearly $x \geqslant 2$ implies $f_n(x) \geqslant f_n(2)$ so it suffices to show $f_n(2 - 1/2^{n-1}) < 0$ to prove the lemma.

$$f_n(2 - 2^{1-n}) = (2 - 2^{1-n})^n + (1 - (2 - 2^{1-n})^n)/(1 - 2^{1-n})$$

$$= 1 - 2^{1-n}(2 - 2^{1-n})^n \quad \text{divided by } 1 - 2^{1-n} > 0$$

$$= 1 - 2^{1-n}\left(\sum_{i=0}^{n} \binom{n}{i} (-1)^i\, 2^{n - i + i(1-n)}\right)$$

$$= 1 - 2^{1-n}\left(\sum_{i=0}^{n} \binom{n}{i} (-1)^i\, 2^{n(1-i)}\right)$$

Comparing the absolute value of neighbouring terms in the expansion of $(2 - 2^{1-n})^n$;

$$\left(\binom{n}{i} 2^{n(1-i)} \right) / \left(\binom{n}{i+1} 2^{-ni} \right) = 2^n (i+1)/(n-i)$$

$$\geqslant 1 \quad \text{if } n \text{ is large.}$$

Since neighbouring terms in the expansion have opposite signs we may ignore all but the first two to conclude;

$$f_n(2 - 2^{1-n}) \leq 1 - 2^{1-n}(2^n - n)$$

$$= 1 - 2 + n/2^{n-1}$$

$$< 0 \quad \text{if } n \text{ is large.}$$

Section Three.

ON GROUP ACTIONS WITH QUASI-DISCRETE SPECTRUM AND UNIFORM DISTRIBUTION (MOD ONE).

1. Quasi-Discrete Spectrum.

The theory of single transformations with quasi-discrete spectrum has been investigated by Abromov (1) in the measure theoretic category and by Hahn and Parry (10) in the topological. Hahn (12) started an investigation of such a theory for actions of the group $\mathbb{R}$, with the discrete topology. Parry restructured Hahn's theory (20) and considered actions of sub-groups of $\mathbb{R}$. We are mainly interested in topological actions of finitely generated abelian groups, especially $\mathbb{Z}^m$. Throughout this section a group action (or semi-group action) will mean a continuous action of the group (or semi-group) as homeomorphisms (or continuous mappings) of a compact Hausdorff space. A dynamical system will mean the same thing.

The assumption in (20) that the group acting is a sub-group of $\mathbb{R}$ is not, in fact, needed. The proofs go through merely under the assumption that the group is abelian. In this subsection we give an exposition of the theory of discrete abelian group actions with quasi-discrete spectrum as theorems 2,3,4 and 5. The proofs of theorems 2,3 and 4 are almost identical with those of the corresponding results in (20). Our approach to theorem 5 is slightly different and we have been able to show that one of the hypotheses of this theorem is necessary. It is suggested in (12) and (20) that this might not be the case. The demonstration of this necessity is in subsection 4, following our consideration of the problem of showing there are group actions to which the theory applies. Finally, in this subsection, we consider the class of transformations to which a given group action of the type we are considering can be transversal, in the sense of

Sinai (26).

Throughout this section we shall use a number of well known facts about locally compact groups and their dual groups. In particular we shall often make no distinction between a group and its double dual. A full discussion of these topics may be found in (14).

We commence by giving the basic definitions of (12). Let Υ be a discrete abelian group and $T_t : X \to X$ be a homeomorphism of a compact Hausdorff space, for all $t \in \Upsilon$. $g \in \mathcal{C}(X, \mathbb{K})$, where $\mathcal{C}(X, \mathbb{K})$ is the multiplicative group of continuous complex functions on X of absolute value one, is an eigenfunction of the Υ - action if for all $t \in \Upsilon$ there exists $\alpha(t) \in \mathbb{K}$ such that $g(T_t x) = \alpha(t).g(x)$ for all $x \in X$. Notice that;

$$\begin{aligned} \alpha(t + t') &= g_\circ T_{t+t'}/g = (g_\circ T_t/g)_\circ T_{t'} \, . \, (g_\circ T_{t'}/g) \\ &= \alpha(t) \, . \, \alpha(t') \qquad \text{all } t, t' \in \Upsilon . \end{aligned}$$

In other words $\alpha : \Upsilon \to \mathbb{K}$ is a character of Υ. Let $G_1 \subset \mathcal{C}(X, \mathbb{K})$ denote the group of eigenfunctions of the action.

Given groups of elements of $\mathcal{C}(X, \mathbb{K})$; $G_1 \subset G_2 \subset \dots \subset G_{n-1}$ we define;

$$G_n = \left\{ g \in \mathcal{C}(X, \mathbb{K}) : g_\circ T_t/g \in G_{n-1} \quad \text{for all} \quad t \in \Upsilon \right\}.$$

Then G_n is also a sub-group of $\mathcal{C}(X, \mathbb{K})$ and is called the group of quasi eigenfunctions of order n. Let $G = \bigcup_n G_n$ be the group of all quasi eigenfunctions. The Υ - action has quasi-discrete spectrum if the sub-algebra of $\mathcal{C}(X)$ generated by G is dense. Equivalently, since constant functions are trivially eigenfunctions, the action has quasi-discrete spectrum if G seperates points, by the Stone-Wierstrass theorem.

For each $t \in \Upsilon$ and $n \in \mathbb{N}^+$ we also define;

$$H_n^t = \left\{ h \in G_{n-1} : \exists g \in G_n \text{ such that } g \circ T_t / g = h \right\}.$$

where $G_0 = \mathbb{K}$. Finally we define the group of quasi eigenvalues of order n to be;

$$H_n = \left\{ h : \Upsilon \times X \to \mathbb{K} : \forall t \in \Upsilon \ \ h(t, \ ) \in H_n^t \right\}.$$

So $H = \bigcup_n H_n$ is the group of all quasi eigenvalues.

<u>Lemma 1.</u> Suppose Υ is a finitely generated group and the action has generators $T_1, \ldots T_m$. Then the definition of G_n is equivalent to;

$$G_n = \left\{ g \in G(X, \mathbb{K}) : g \circ T_j / g \in G_{n-1} \text{ for all } 1 \leq j \leq m \right\}.$$

Similarly G_1 is the sub-group of $G(X, \mathbb{K})$
$\left\{ g : g \circ T_j / g \in \mathbb{K} \text{ for all } 1 \leq j \leq m \right\}$.

<u>Proof.</u> We first note another simple fact which does not depend on the hypothesis of being finitely generated.

If $g \in G_n$ then for all $t \in \Upsilon$; $g \circ T_t \in G_n$ also.

We prove this by induction on n.

If $n = 1$; $\quad g \circ T_t \circ T_{t'} / g \circ T_t = \alpha(t') \circ T_t$
$$= \alpha(t') \text{ for all } t' \in \Upsilon.$$

I.e. $g \circ T_t \in G_1$.

Let $g \in G_n$ so for all $t' \in \Upsilon$ there exists $h_{t'} \in G_{n-1}$ such that $g \circ T_{t'} / g = h_{t'}$.

Then for all $t' \in \mathcal{T}$;

$$g_\circ T_{t\circ} T_{t'}/g_\circ T_t \quad = \quad h_{t'\circ} T_t \quad \in \quad G_{n-1}\,,$$ by the induction hypothesis.

I.e. $g_\circ T_t \in G_n$.

To prove the lemma we show that if $g_\circ T_t/g \in G_{n-1}$ and $g_\circ T_{t'}/g \in G_{n-1}$ then we have;

(1). $g_\circ T_{-t}/g \in G_{n-1}$ and (2). $g_\circ T_{t+t'}/g \in G_{n-1}$.

(1). $g_\circ T_t \;=\; h_t . g$ so $g_\circ T_{-t}/g \;=\; 1/h_{t\circ} T_{-t} \;\in\; G_{n-1}$.

(2). $g_\circ T_{t+t'}/g \;=\; (h_t . g)_\circ T_{t'}/g \;=\; h_{t'} . h_{t\circ} T_{t'} \;\in\; G_{n-1}$.

We refer the reader to Ellis's notes (5) for the definition of a distal flow and the properties of such actions.

<u>Proposition 1.</u> A $\mathcal{T}$- action with quasi-discrete spectrum is distal.

<u>Proof.</u> We suppose there exist $x,y,z \in X$ and a net t_i such that;

$$T_{t_i}(x) \longrightarrow y \qquad\qquad T_{t_i}(z) \longrightarrow y$$

Then we must show $x = z$ to prove the proposition.
Since all elements of G are continuous;

$$g(T_{t_i}(x)) \longrightarrow g(y) \qquad\qquad g(T_{t_i}(z)) \longrightarrow g(y)$$

for all $g \in G$ and we show that $g(x) = g(z)$ if $g \in G$.

Let $g \in G_1$ so $g \circ T_t = \alpha(t).g$. Thus;

$$g(x).\alpha(t_i) \longrightarrow g(y) \qquad g(z).\alpha(t_i) \longrightarrow g(y)$$

Hence $g(x) = g(z)$.

Suppose, as an induction hypothesis, $g(x) = g(z)$ for all $g \in G_{n-1}$.

Then if $g \in G_n$ so $g \circ T_t = h_t . g$, where $h_t \in G_{n-1}$;

$$g(x).h_{t_i}(x) \longrightarrow g(y) \qquad g(z).h_{t_i}(z) \longrightarrow g(y)$$

so $g(x) = g(z)$.

We shall impose a further restriction on the group actions we consider. The following assumption should be understood to be implicit in all the following results. For all $f \in \mathcal{C}(X)$ the existance of a sub-group $\tilde{\mathcal{J}}'$ of $\tilde{\mathcal{J}}$ such that;

(1) $|\tilde{\mathcal{J}}/\tilde{\mathcal{J}}'| < \infty$

(2) $f \circ T_t = f$ for all $t \in \tilde{\mathcal{J}}'$.

implies that f is constant. We refer to this assumption by the designation (A). A sub-group which has property (1) is often called <u>syndetic.</u>

Assumption (A) is implied by the group action being totally minimal or totally ergodic with respect to a measure which is positive on open sets. Notice also that (A) implies that if $g, g' \in G$ and these two quasi eigenfunctions correspond to the same quasi eigenvalue, i.e.;

$$g \circ T_t / g \quad = \quad h_t \quad = \quad g' \circ T_t / g' \qquad \text{for all } t \in \tilde{\mathcal{J}},$$

then g is a constant multiple of g'.

Let $g \in G_1$ so $g \circ T_t = \alpha(t).g$. Thus;

$$g(x).\alpha(t_i) \longrightarrow g(y) \qquad g(z).\alpha(t_i) \longrightarrow g(y)$$

Hence $g(x) = g(z)$.

Suppose, as an induction hypothesis, $g(x) = g(z)$ for all $g \in G_{n-1}$.

Then if $g \in G_n$ so $g \circ T_t = h_t.g$, where $h_t \in G_{n-1}$;

$$g(x).h_{t_i}(x) \longrightarrow g(y) \qquad g(z).h_{t_i}(z) \longrightarrow g(y)$$

so $g(x) = g(z)$.

We shall impose a further restriction on the group actions we consider. The following assumption should be understood to be implicit in all the following results. For all $f \in \mathcal{C}(X)$ the existence of a sub-group $\mathcal{T}'$ of $\mathcal{T}$ such that;

(1) $|\mathcal{T}/\mathcal{T}'| < \infty$

(2) $f \circ T_t = f$ for all $t \in \mathcal{T}'$.

implies that f is constant. We refer to this assumption by the designation (A). A sub-group which has property (1) is often called <u>syndetic.</u>

Assumption (A) is implied by the group action being totally minimal or totally ergodic with respect to a measure which is positive on open sets. Notice also that (A) implies that if $g, g' \in G$ and these two quasi eigenfunctions correspond to the same quasi eigenvalue, i.e.;

$$g \circ T_t / g = h_t = g' \circ T_t / g' \qquad \text{for all } t \in \mathcal{T},$$

then g is a constant multiple of g'.

<u>Lemma 2.</u> Let F be a sub group of $\mathcal{C}(X, \mathbb{K})$ which contains all the constant functions and is invarient under composition by T_t, for all $t \in \mathcal{J}$. Suppose m is a Baire probability measure on X such that $\int_X f \, dm = 0$ if $f \in F$ is not constant. Consider an element, g, of $\mathcal{C}(X, \mathbb{K})$ of the form $g = \sum_{i=1}^{\infty} c_i . f_i$, where $c_i \in \mathbb{C}$ and $f_i \in F$ for all i. Then if for all $t \in \mathcal{J}$ there exists $f_t \in F$ such that $g \circ T_t = f_t . g$ we conclude $g \in F$.

<u>Proof.</u> We consider orthogonality in $L^2(m)$.

Since, by assumption, pairs of elements of F are either orthogonal or constant multiples of each other we may assume without loss of generality that in the expansion of g $f_i \perp f_j$ unless $i = j$. For all t;

$$\sum c_i . f_i \circ T_t \quad = \quad g \circ T_t \quad = \quad \sum c_i . f_t . f_i \qquad (*)$$

By assumption on F we see;

$$f_t . f_i \quad = \quad c_{i,t} \cdot f_{r_t(i)} \circ T_t ,$$

where $c_{i,t} \in \mathbb{K}$ and r_t is a permutation of $\{ i : c_i \neq 0 \}$.

Now let $i = r_s(j)$ so;

$$\begin{aligned} c_{i,t} . f_{r_t(i)}(T_t x) \quad &= \quad f_t(x) . f_i(x) \\ &= \quad f_t(x) . \left(f_s(T_{-s} x) . f_j(T_{-s} x) \,/\, c_{j,s} \right) \end{aligned}$$

In other words;

$$\begin{aligned} c_{i,t} . c_{j,s} . f_{r_t(r_s(j))}(T_{t+s} x) \quad &= \quad f_t(T_s x) . f_s(x) . f_j(x) \\ &= \quad f_{t+s}(x) . f_j(x) \end{aligned}$$

The last expression equals $c_{j,t+s}\cdot f_{r_{t+s}(j)}(T_{t+s}x)$ so we conclude $r_t \circ r_s = r_{t+s}$.

Now choose some i such that $c_i \neq 0$ and fix it for the rest of the proof.

$\{t \in \mathcal{T} : r_t(i) = i\} = B_i$ is a sub group of $\mathcal{T}$.

Since the number of coefficients, c_j, such that $|c_j| = |c_i|$ must be finite we see, from (*) and the uniqueness of the expansion of g, that the set $\{r_t(i) : t \in \mathcal{T}\}$ is finite.

Thus B_i is a syndetic subgroup of $\mathcal{T}$.

If $t \in B_i$ then, from (*), notice that $c_{i,t} = 1$ so;

$$f_i(T_t x)\,/f_i(x) \quad = \quad f_t(x) \quad = \quad g(T_t x)\,/\,g(x)\ ,$$

for all $t \in B_i$.

(A) implies f_i/g is constant and the lemma is proved.

<u>Theorem 2.</u> A $\mathcal{T}$ - action with quasi-discrete spectrum has a unique invarient Baire probability measure. In fact if m is any invarient ergodic Baire probability then $\int_X g\, dm = 0$ if $g \in G$ is not constant.

<u>Proof.</u> The first statement follows from the second since the algebra generated by G is dense in $\mathcal{C}(X)$.

We prove the second statement by induction on n, where $g \in G_n$.

If $g \in G_1$ then g not a constant implies, by (A), there exists $t \in \mathcal{T}$ such that $g \circ T_t \,/\, g = \alpha(t) \neq 1$.

Then;

$$\int_X g\, dm \quad = \quad \int_X g\circ T_t\, dm \quad = \quad \alpha(t)\,.\int_X g\, dm$$

So $\int_X g \, dm = 0$.

Suppose the result is true for all $g \in G_{n-1}$.

Recall, from the proof of lemma 1, that G_{n-1} is invarient under composition by T_t, for all t.

Thus if $\mathcal{A}$ is the smallest σ-algebra with respect to which all elements of G_{n-1} are measurable then $\mathcal{A}$ is also invarient under the group action.

We let $g \in G_n$ so $g \circ T_t = h_t . g$, where $h_t \in G_{n-1}$ for all t.

Now;

$$E(g / \mathcal{A}) \circ T_t = E(g \circ T_t / \mathcal{A}) = E(h_t . g / \mathcal{A})$$
$$= h_t . E(g / \mathcal{A})$$

$E(g / \mathcal{A})$ is a constant multiple of g (mod 0), by ergodicity.

Either $\int_X E(g / \mathcal{A}) \, dm = \int_X g \, dm = 0$ or the constant is one.

In the second case g is measurable with respect to $\mathcal{A}$.

Let Y be the space consisting of all points of the form;

$$y = \bigcap_{f \in G_{n-1}} f^{-1}(k_f), \text{ where } k_f \in \mathbb{K} \text{ for all } f.$$

Then Y is naturally a compact Hausdorff factor space of X and $\mathcal{A}$ is a sub σ-algebra of the Baire sets of Y.

Thus $g \in \mathcal{C}(Y)$ and, by the Stone Wierstrass theorem, we may apply lemma 2 with $F = G_{n-1}$.

This completes the proof.

We use theorem 2 in arguments later on so we have given a full proof of this result. None of our main results depend in a formal way on any subsequent theorems in this subsection so we shall only outline the proofs of the remaining theorems in our exposition.

Lemma 3. If $g \in G$ and there exists a positive integer p such that $g^p = 1$ then $g = 1$.

Proof outline. By induction on the minimum order of g.

Theorem 3. A $\mathcal{T}$ - action with quasi-discrete spectrum is topologically conjugate to an action of $\mathcal{T}$ as affine transformations of a compact, connected abelian group.

Proof outline. $G = \mathbb{K}.\Gamma$, where Γ is a sub group of G

such that $\Gamma \cap \mathbb{K} = \{1\}$.

Then $\hat{\Gamma} = Y$ is a compact, connected abelian group.

G is invarient under the group action so for all $\gamma \in \Gamma$;

$\gamma \circ T_t = s_t(\gamma).\hat{S}_t(\gamma)$.

Here $s_t \in Y$ and $\hat{S}_t$ is an automorphism of Γ.

The group of affines is $\{s_t.S_t : Y \to Y\}$.

The existence of a topological conjugacy is equivalent to the existence of a Banach algebra isomorphism from $\mathcal{C}(Y)$ to $\mathcal{C}(X)$ which commutes with the group actions.

This follows from proposition 1 and theorem 2.

Corollary. The conjugacy is also a measure theoretic one, with respect to the only invarient measures.

The group of quasi eigenfunctions of the conjugate transformations constructed in theorem 3 is $\mathbb{K}.\hat{Y}$. Since $\hat{Y}$ is canonically isomorphic to Γ we may regard the two conjugate actions as having the same group of quasi eigenfunctions.

For all $t \in \mathcal{T}$ define a homomorphism $\sigma_t : \Gamma \to \Gamma$ by $\sigma_t(g) = g{\circ}S_t / g$. The remark at the bottom of page 3-1-5 shows that the map $\tau : \Gamma \to H$ defined by $\tau(g) = g{\circ}T_t / g$ is an isomorphism and, further, $\tau : G_n \cap \Gamma \to H_n$ is also an isomorphism. Thus we may define homomorphisms $\tilde{\sigma}_t = \tau.\sigma_t.\tau^{-1}$ of H and $\tilde{s}_t = s_t.\tau^{-1} \in \hat{H}$.

Recall that one may regard H_1 as a sub group of $\hat{\mathcal{T}}$. In fact if $g \in G_1 \cap \Gamma$ then;

$$\tau(g) = g(s_t).g{\circ}S_t / g = g(s_t) = \tilde{s}_t(\tau(g)) = \alpha(t).$$

In other words $\tau(g)$ is the map $t \mapsto g(s_t)$.

One may derive a number of properties of the group H associated with a group action of the type we are considering. It is often easiest to derive properties of the isomorphism class of H and H_n by considering Γ and $G_n \cap \Gamma$. We list these properties in the following definition.

<u>Definition.</u> An <u>abstract system of quasi eigenvalues</u> for the group $\mathcal{T}$ is a sequence $\hat{\mathcal{T}} \supset A_1 \subset A_2 \subset A_3 \subset \ldots$ of discrete, torsion free abelian groups and, for all $t \in \mathcal{T}$, homomorphisms $\tilde{\sigma}_t : A \to A$, where $A = \bigcup_n A_n$, with the following properties;

(1) $A_1 = \bigcap_t \ker(\tilde{\sigma}_t)$

(2) for all t; $\tilde{\sigma}_t(A_n) \subset A_{n-1}$ and $A_n = \bigcap \ker(\tilde{\sigma}_{t_1} \circ \ldots \tilde{\sigma}_{t_n})$

where the intersection is over all $(t_1, \dots t_n) \in \mathcal{T}^n$.

(3) for all $a \in A$ and $t,u \in \mathcal{T}$;

$$\tilde{\sigma}_u(\tilde{\sigma}_t(a)) = \tilde{\sigma}_{u+t}(a) / \tilde{\sigma}_u(a).\tilde{\sigma}_t(a) = \tilde{\sigma}_t(\tilde{\sigma}_u(a))$$

(4) for all t there exist characters $\tilde{s}_t \in \hat{A}$ such that;

(i) if $a \in A_1$; $\tilde{s}_t(a) = a(t)$

(ii) for all $a \in A$; $\tilde{s}_{t+u}(a) = \tilde{s}_t(a).\tilde{s}_u(a).\tilde{s}_u(\tilde{\sigma}_t(a))$.

<u>Theorem 4.</u> Given an abstract system of quasi eigenvalues for $\tilde{\mathcal{T}}$ there is a uniquely ergodic action of $\tilde{\mathcal{T}}$ as affine transformations of a compact connected group, Y, such that $G = \mathbb{K}.\hat{Y}$ and $\hat{Y} \cap G_n = A_n$.

<u>Proof outline.</u> Consider $Y = \hat{A}$.

Let $\hat{\sigma}_t : Y \to Y$ be dual to $\tilde{\sigma}_t$ and define $T_t : Y \to Y$ by $y \mapsto \tilde{s}_t.y.\hat{\sigma}_t(y)$.

Clearly all elements of A_n are quasi eigenfunctions of order n.

If the action is totally ergodic, with respect to Haar measure, then it will satisfy hypothesis (A) of theorem 2. This would imply, by theorem 2, that we have identified all quasi eigen functions and the action is uniquely ergodic. In fact the action is totally ergodic as may be shown by considering the fourier series of an invarient L^2 function and using the following fact;

If $a \circ T_t = a$ for all t in a syndetic sub group of $\tilde{\mathcal{T}}$ then a is the identity element.

A natural question to consider is when two actions (of the same group) of the type we are considering are topologically conjugate. It is easy to see that any conjugacy must be an affine, in an obvious sense, map. The two conditions in the next theorem may be easily seen to be necessary.

Theorem 5. Let $\{T_t : X \to X\}$ and $\{T'_t : X' \to X'\}$ be Υ - actions with quasi discrete spectrum. Suppose that;

(1) there exists an isomorphism $V : H' \to H$ such that $\tilde{\sigma}_t \circ V = V \circ \tilde{\sigma}'_t$, for all t.

(2) there exists $v \in \hat{H'}$ such that $\tilde{s}_t \circ V / \tilde{s}'_t = v \circ \tilde{\sigma}'_t$, for all t.

Then the two group actions are conjugate.

Proof outline. The conjugacy is the map $v.V : X \to X'$, if we regard the group actions as represented in the manner described by theorem 3.

The proof depends on two assertions;

(i). The process of taking an action as represented by theorem 3, regarding H as a system of abstract quasi eigenvalues and then applying theorem 4 yields a action which is topologically conjugate to the first.

(ii). Applying theorem 4 to H and H' which satisfy the conditions of this theorem yields conjugate actions.

The proof of (i) follows from the proof of theorem 4.
The conjugacy is the map $\hat{\tau} : \hat{H} \to \hat{\Gamma}$.

(ii) may be shown by direct verification.

Lemma 4. Conditions (1) and (2) of theorem 5 imply;

(3) $V : H_1' \longrightarrow H_1$ is the identity isomorphism if we regard H_1' and H_1 as sub groups of $\hat{\mathcal{J}}$.

Proof. From (1) it is clear that $V : H_1' \longrightarrow H_1$ is an isomorphism.

If $h' \in H_1'$ then for all t;

$$\tilde{s}_t \circ V(h') \,/\, \tilde{s}_t'(h') \;=\; v_\circ \tilde{\sigma}_t'(h') \;=\; 1, \quad \text{by (1) on p. 3-1-10}$$

So, recalling (4), (i) on p. 3-1-11, for all t;

$$V(h')(t) \;=\; \tilde{s}_t(V(h')) \;=\; \tilde{s}_t'(h') \;=\; h'(t).$$

I.e. $V(h') = h'$.

If $\mathcal{J}$ is the group $\mathbb{Z}$ then, as is shown in (10), (2) may be deduced from (1) and (3). In subsection 4 we give an example, where $\mathcal{J}$ is the group $\mathbb{Z}^2$, which shows hypothesis (2) is necessary in the sense that it is not implicit in (1) and (3). Mary Reese has told me that she also knows this.

The notion of group actions being transversal, a generalisation of commuting, has been exploited by, in particular, Sinai (26). See also Kowada (16). The next result may be interpreted as saying that a continuous map to which a group action of the type we have been considering is transversal, is topologically conjugate to an affine transformation of a compact, connected abelian group.
The next theorem does not depend on our usual assumption (A).

Theorem 6. Let $F : Y \to Y$ be a continuous map on a compact abelian group and $T_t : Y \to Y$ be a $\mathcal{T}$ - action with quasi-discrete spectrum such that $G = \mathbb{K}.\hat{Y}$. Suppose there exists a homomorphism $f : \mathcal{T} \to \mathcal{T}$ such that either;

(1) $F_\circ T_t = T_{f(t)} \circ F$ for all $t \in \mathcal{T}$.

(2) $F_\circ T_{f(t)} = T_t \circ F$ for all t and f is onto.

Then F is an affine transformation of Y.

Proof. Let $U_F(g) = g_\circ F$ and $U_t(g) = g_\circ T_t$.

We prove that U_F maps G to G in case (1).

The same fact can be proved in case (2) by a similar argument.

Let $g \in G_1$ so $U_t(g) = \alpha(t).g$ and;

$$U_t(U_F(g)) = U_F(U_{f(t)}(g)) = \alpha(f(t)).U_F(g).$$

for all $t \in \mathcal{T}$.

I.e. $U_F(g)$ is an eigenfunction.

Assume U_F maps G_{n-1} to G_{n-1} and let $g \in G_n$.

Then for all t there exists $h_t \in G_{n-1}$ such that $U_t(g) = h_t.g$ and so;

$$U_t(U_F(g)) = U_F(U_{f(t)}(g)) = U_F(h_{f(t)}).U_F(g) .$$

I.e. $U_F(g) \in G_n$ and the claim is true, by induction.

Now we know U_F maps $\mathbb{K}.\hat{Y}$ to $\mathbb{K}.\hat{Y}$ and the rest of the proof is standard.

There is a homomorphism $E : \hat{Y} \to \hat{Y}$ and $y \in Y$ such that for all $\gamma \in \hat{Y}$; $U_F(\gamma) = y(\gamma).E(\gamma)$.

Then $F = y.\hat{E}$

<u>Corollary.</u> Let $\{T_t : X \to X\}$ be a $\mathcal{T}$- action which satisfies the hypotheses of theorem 3. Let $\{F_s : X \to X\}$ be a semi group action such that for all s there exists a homomorphism f_s of $\mathcal{T}$ which satisfies (1) or (2). Then the semi group action is topologically conjugate to an action as affine transformations of a compact connected abelian group.

<u>Proof.</u> Let $\theta : X \to Y$ be the conjugacy given by theorem 3.

Consider, for each s, $\theta \circ F_s \circ \theta^{-1}$.

The additional hypothesis in case (2) that f is an onto homomorphism may seem unnatural but it is needed. Let Δ be the 2 - adic integers and $d = (\ldots 0, \ldots 0, 1) \in \Delta$. Consider the transformation $\delta : \Delta \to \Delta$ defined by $\delta(x) = x + d$. Then δ is an ergodic translation of a compact abelian group and, as in (13), has discrete spectrum with eigenfunctions $\mathbb{K}.\hat{\Delta}$. This system was christened the "adding machine" by Furstenberg who also pointed out that if Σ is the (one-sided) shift on Δ then $\Sigma \circ \delta^2 = \delta \circ \Sigma$. However Σ is certainly not affine. This Z action does not satisfy condition A.

2. Background to the Rest of Section 3.

In 1916 H. Weyl proved the following beautiful theorem.

Theorem. (Weyl) A polynomial of finitely many integer variables with real coefficients is uniformly distributed (mod one) if and only if at least one coefficient, other than the constant term, is irrational.

F. Hahn (11) has given an ergodic theoretic proof of this result in the special case of one variable by showing certain affine transformations of finite dimensional tori are uniquely ergodic. Our main aim is to give a similar proof of the general case by constructing actions of finitely generated abelian groups as affine transformations of finite dimensional tori. These actions will satisfy all the hypotheses of theorems 2 and 3 and are already represented in the way described by theorem 3.

A simple consequence of our construction is that we can show there are many $\mathbb{Z}^m$ - actions to which the theory outlined in subsection 1 applies. This may also be seen quite easily by more direct methods.

In the remainder of this subsection we show, in a very special case, how to construct a dynamical system of the required type from a polynomial. This is intended to motivate the general construction in subsection 3, where we give a formal definition without refering to any polynomials.

~~we give in subsection 5 does not, however, reduce to that of Hahn in the special case he considered.~~

~~The motivation for considering the particular construction in subsection 3 is derived from properties of certain groups of functions of integer variables taking values in the circle group;~~ ~~$\mathbb{K}$. The discussion is limited to a very special case which suffices to indicate the direction to be taken in the next subsection.~~

$\Upsilon(p_1,p_2) \;=\; \sum a(i_1,i_2)\, p_1^{i_1}\, p_2^{i_2} + a(0)$ will denote a fixed real valued polynomial of degree two in two integers variables. Thus the sum is over all $(i_1,i_2) \in \mathbb{N}^2$ such that $1 \leq i_1 + i_2 \leq 2$. We place the restriction on the real numbers; $\{a(i_1,i_2)\}$ that there exist no $(Q,Q') \in \mathbb{Z}^2$, apart from $(0,0)$, with the properties;

$$\begin{aligned} &2.Q.a(2,0) + Q'.a(1,1) \in \mathbb{Z}. \\ &2.Q'.a(0,2) + Q.a(1,1) \in \mathbb{Z}. \end{aligned} \qquad (*)$$

We denote a general polynomial of degree two in two integer variables with real coefficients $f : \mathbb{Z}^2 \to \mathbb{R}$ as follows;

$$f(p_1,p_2) \;=\; \sum b(i_1,i_2)\, p_1^{i_1}\, p_2^{i_2} + b(0),$$

summing as above. If $\exp : \mathbb{R} \to \mathbb{K}$ is the map $\exp(r) = e^{2\pi i r}$ then for each $j_1, j_2 \in \mathbb{Z}$ let;

$$U(j_1,j_2)(\exp f(p_1,p_2)) \;=\; \exp f(p_1+j_1, p_2+j_2).$$

Then we see that in fact this is equal to;

$$\exp\Big(\sum b(i_1,i_2)\, p_1^{i_1}\, p_2^{i_2} + b(0) + \sum b(i_1,i_2)\, j_1^{i_1}\, j_2^{i_2} + 2j_1 b(2,0)p_1 + 2j_2 b(0,2)p_2 + j_1 b(1,1)p_2 + j_2 b(1,1)p_1\Big).$$

We can rewrite $U(j_1, j_2)(\exp f(p_1, p_2))$ =

$$\exp\big(f(p_1,p_2) + f(j_1,j_2) - b(0) + (2j_1 b(2,0) + j_2 b(1,1))p_1 + (2j_2 b(0,2) + j_1 b(1,1))p_2 \big).$$

Let $V(j_1, j_2)(\exp f) = U(j_1, j_2)(\exp f) / \exp f$. Then we see that for all $j_1, j_2 \in \mathbb{Z}$; $V(j_1, j_2)(\exp f)$ is the exponential of a polynomial of degree only one. Further for all $j_1, j_2, j_1', j_2' \in \mathbb{Z}$ we have;

$$V(j_1, j_2)\big(V(j_1', j_2')(\exp f) \big) \in \mathbb{K}.$$

It appears that the group of transformations; $\{ U(j_1, j_2) : (j_1, j_2) \in \mathbb{Z}^2 \}$ has certain quasi-discrete spectrum like properties. Let G denote the smallest sub-group of $\mathcal{C}(\mathbb{R}^{\mathbb{Z}^2}, \mathbb{K})$ containing $\exp \Upsilon$ and closed under the transformations $U(j_1, j_2)$ and also closed under multiplication by elements of $\mathbb{K}$. The idea (recalling the proof of theorem 3) is to regard G as $\mathbb{K}.\hat{X}$. Having discovered X the point transformations induced by the transformations $U(j_1, j_2)$ should have quasi-discrete spectrum. Let Γ denote the following sub-~~group~~ set of $\mathcal{C}(\mathbb{R}^{\mathbb{Z}^2}, \mathbb{K})$;

$$\Big\{ \exp f : f(p_1,p_2) = \sum_{i=1}^{L} \gamma_i (\Upsilon(p_1,p_2) - a(0)) + \sum_{i=1}^{L} \gamma_i \cdot j_{1,i} (2a(2,0)p_1 + a(1,1)p_2) + \sum_{i=1}^{L} \gamma_i \cdot j_{2,i} (2a(0,2)p_2 + a(1,1)p_1), \text{ for some } L \in \mathbb{N},$$
$$\gamma_i = \pm 1 \text{ and } (j_{1,i}, j_{2,i}) \in \mathbb{Z}^2, \text{ where } 1 \leq i \leq L \Big\}.$$

Let $k_1 = \sum_{i=1}^{L} \gamma_i$, $k_2 = \sum_{i=1}^{L} \gamma_i \cdot j_{1,i}$, $k_3 = \sum_{i=1}^{L} \gamma_i \cdot j_{2,i}$.

Then we can write a typical element of Γ in the form;

$$\exp\left(k_1.(\Upsilon(p_1,p_2) - a(0)) + k_2.(2a(2,0)p_1 + a(1,1)p_2) + k_3.(2a(0,2)p_2 + a(1,1)p_1) \right).$$

<u>Lemma 1.</u> Γ is a sub-group of G and further;

(1). $\mathbb{K}.\Gamma = G$.

(2). $\Gamma \cap \mathbb{K} = \{1\}$.

(3). Γ is isomorphic to $\mathbb{Z}^3$.

<u>Proof.</u> All these statements are clear except (3), which we now prove.

We claim that the isomorphism is $\exp(f) \longmapsto (k_1,k_2,k_3)$, where $\exp(f)$ is the typical element of Γ described by the expression at the top of this page.

Clearly the map is a homomorphism.

Suppose $\exp(f) = 1$. We show that in this case $(k_1,k_2,k_3) = 0$ and hence the map is well defined and one to one.

$$(\exp f(1,1))/(\exp(f(1,0) + f(0,1))) = \exp(k_1.a(1,1))$$

By supposition this is one so we conclude $a(1,1)$ is rational if $k_1 \neq 0$.

(*) implies at least one of the numbers; $a(2,0)$, $a(1,1)$ and $a(0,2)$ is irrational.

However;

$$\exp f(2,0) / \exp 2.f(1,0) = \exp k_1.2a(2,0) = 1$$

We conclude $a(2,0)$ is rational if $k_1 \neq 0$ and, similarly, $a(0,2)$ is rational if $k_1 \neq 0$.

Thus we must have $k_1 = 0$.

In this case;

$$\exp f(1,0) = \exp(k_2.2a(2,0) + k_3.a(1,1)) = 1$$

$$\exp f(0,1) = \exp(k_3.2a(0,2) + k_2.a(1,1)) = 1$$

which is precisely what is excluded by (*) unless $k_2 = k_3 = 0$.

It only remains to show the map is onto.
Given any $(k_1,k_2,k_3) \in \mathbb{Z}^3$ define L, γ_i, $j_{1,i}$ and $j_{2,i}$ as follows;

$$L = k_1 + 2.$$

$$\gamma_1 = 1, \quad \gamma_2 = -1, \quad \gamma_i = \begin{cases} 1 & \text{if } k_1 \geq 0 \\ -1 & \text{if } k_1 \leq 0 \end{cases}$$

$$j_{1,1} = k_2, \quad j_{2,1} = k_3, \quad j_{1,i} = j_{2,i} = 0 \quad \text{if } 2 \leq i \leq L.$$

The integers thus defined satisfy the defining equations for the triple (k_1,k_2,k_3) at the bottom of page 3-2-2 and we conclude the map is indeed onto.

$X = \hat{\Gamma}$ is isomorphic to $\mathbb{K}^3$. $U(j_1,j_2)$ does not map Γ to Γ but Γ to $\mathbb{K}.\Gamma$. We define $s(j_1,j_2) \in X$ and automorphisms $\hat{S}(j_1,j_2)$ of Γ by;

$$U(j_1,j_2)(\exp f) = s(j_1,j_2)(\exp f) \,.\, \hat{S}(j_1,j_2)(\exp f)$$

Then there are automorphisms $S(j_1,j_2) : X \to X$, dual to $\hat{S}(j_1,j_2)$. Let $T(j_1,j_2) = s(j_1,j_2).S(j_1,j_2)$ be an affine transformation of X, for each $(j_1,j_2) \in \mathbb{Z}^2$.

From the equation at the top of page 3-2-3 we observe that;

$$s(j_1,j_2)(\exp f) = \exp f(j_1,j_2)$$

since Γ consists of precisely those elements, exp f, of G such that $\exp f(0,0) = 1$.

As a character of $\mathbb{Z}^3$;

$$s(j_1,j_2)(k_1,k_2,k_3) = \exp(\, k_1.(\gamma(j_1,j_2) - a(0)) + k_2.(2a(2,0)j_1 + a(1,1)j_2) + k_3.(2a(0,2)j_2 + a(1,1)j_1\,),$$

by the expression at the top of page 3-2-4.

In other words we can regard $s(j_1,j_2)$ as the element of $\mathbb{K}^3$;

$$(\, \exp(\gamma(j_1,j_2) - a(0)),\ \exp(2a(2,0)j_1 + a(1,1)j_2),\ \exp(2a(0,2)j_2 + a(1,1)j_1)\,).$$

Refering to the equation at the top of page 3-2-3 we see;

$$\hat{S}(j_1,j_2)(\exp f) = \exp(\, f(p_1,p_2) + (2j_1b(2,0) + j_2b(1,1))p_1 + (2j_2b(0,2) + j_1b(1,1))p_2\,)$$

If exp f is the element of Γ mapped to (k_1,k_2,k_3) then, from the expression at the top of page 3-2-4, we see $b(i_1,i_2) = k_1.a(i_1,i_2)$ provided $i_1 + i_2 = 2$.

Thus we have;

$$\hat{S}(j_1,j_2)(k_1,k_2,k_3) = (k_1,\ k_2 + j_1.k_1,\ k_3 + j_2.k_1)$$

And for any $(x_1,x_2,x_3) \in \mathbb{K}^3$;

$$S(j_1,j_2)(x_1,x_2,x_3) = (x_1.x_2^{j_1}.x_3^{j_2},\ x_2,\ x_3).$$

The two generators T(1,0) and T(0,1) are therefore given by the equations;

$$T(1,0)(x) = (x_1.x_2.\exp(a(2,0) + a(1,0) - a(0)),\ x_2.\exp(2a(2,0)),\ x_3.\exp(a(1,1)\,)$$

$$T(0,1)(x) = (x_1.x_3.\exp(a(0,2) + a(0,1) - a(0)),\ x_2.\exp(a(1,1)),$$
$$x_3.\exp(2.a(0,2))\ ).$$

Clearly it is simpler to replace $a(2,0) + a(1,0) - a(0)$, $2a(2,0)$, $a(0,2) + a(0,1) - a(0)$ and $2a(0,2)$ by $a'(1,0)$, $a'(2,0)$, $a'(0,1)$ and $a'(0,2)$ respectively. The reader is also warned that we reverse the order of co-ordinates of $\mathbb{K}^3$ in the next subsection.

The construction in subsection 3 is a generalisation of the one just described. We can omit the restriction (*) and just retain the weaker hypothesis that one of the real numbers concerned in the construction is irrational. For this, and other reasons, we also have to restrict the space on which the transformations act to a minimal set. In formal terms we continue in the next subsection without regard for the discussion above.

3. Construction of the Dynamical Systems.

We can construct many different dynamical systems, depending on the choice of a set of real numbers and its orderings. This subsection is devoted to describing the construction of any particular one of them.

Fix some $n \in \mathbb{N}^+$ and $m \in \mathbb{N}^+$. An ordered m'tple of non-negative integers $(i_1, \dots i_m)$ is <u>allowed</u> if $1 \leq \sum_{\lambda=1}^{m} i_\lambda \leq n$. We add such m'tples co-ordinatewise. Consider some fixed set of real numbers at least one of which is irrational;

$$\left\{ a(i_1, \dots i_m) : (i_1, \dots i_m) \text{ is allowed} \right\}.$$

Notation

$$I_j = \left\{ (i_1, \dots i_m) : (i_1, \dots i_m) \text{ is allowed and } i_j \neq 0 \right\}.$$

$$N = |I_j|, \text{ for any } j.$$

Each set I_j is given the lexicographical ordering defined by;

$$\left\{ \left(-\sum_{\lambda=1}^{m} i_\lambda, i_1, \dots i_m\right) : (i_1, \dots i_m) \in I_j \right\}.$$

In other words the elements of I_j are ordered first by the decreasing size of the sum of i_λ's, secondly by the increasing size of i_1's, thirdly by the increasing size of i_2's and so on. $\phi_j : \{1, \dots N\} \to I_j$ is the order preserving bijection.

$$I = \left\{ (i_1, \dots i_m) : i_\lambda \in \mathbb{N},\ 0 \leq \sum_{\lambda=1}^{m} i_\lambda \leq n-1 \right\}.$$

I is given the lexicographical ordering defined by;

$$\left\{ \left(-\sum_{\lambda=1}^{m} i_\lambda , i_1, \dots i_m\right) : (i_1, \dots i_m) \in I \right\}.$$

$\underline{j}$ is the m'tple $(i_1, \dots i_m)$, where $i_j = 1$ and the other co-ordinates are zero. Then if q is a positive integer $q.\underline{j}$ is the m'tple $(0, .. 0, q, 0, .. 0)$ with j'th co-ordinate q.

<u>Lemma 1.</u> $|I| = N$. Let $\phi : \{1, \dots N\} \longrightarrow I$ be the order preserving bijection. Then for any j; $\phi_j(k) = \phi(k) + \underline{j}$, for all $k \in \{1, \dots N\}$. Hence for any j_1 and j_2; $\phi_{j_1}(k) = \phi_{j_2}(k) + \underline{j}_1 - \underline{j}_2$.

<u>Proof.</u> We actually show that the map $\theta : I \longrightarrow I_j$ defined by;

$$\theta(i_1, \dots i_m) = (i_1, \dots i_m) + \underline{j}$$

is an order preserving bijection.

Clearly θ is a bijection, so $|I| = N$, and, further;

$$\left\{ (i_1, \dots i_m) \in I_j : i_1 + \dots + i_m = \lambda \right\} =$$
$$\left\{ \theta(i_1, \dots i_m) : (i_1, \dots i_m) \in I,\ i_1 + \dots + i_m = \lambda - 1 \right\}.$$

Call this set A_λ ~~and~~ let $B_{\lambda-1} = \theta^{-1}(A_\lambda)$ ~~respectively so $\theta(B_{\lambda-1}) = A_\lambda$~~.

For any $1 \leq \lambda \leq n$ the least element of A_λ is $(\lambda-1).\underline{m} + \underline{j}$ and the least element of $B_{\lambda-1}$ is $(\lambda-1).\underline{m}$.

Assume, now, that $(i_1, \dots i_m) \in B_{\lambda-1}$ is not the greatest such element.

The next element in $B_{\lambda-1}$ is;

$(i_1, .. i_{p-1}, i_p+1, i_{p+1}-1, i_{p+2}, .. i_m),$

where p is the greatest integer such that $i_{p+1} \neq 0$.
In A_λ the next element after $\theta(i_1, ... i_m)$ is;

$(i_1, ... i_m) + \underline{p} - \underline{p+1} + \underline{j}$

Thus θ is indeed order preserving.

Notation.

$Q_\lambda = \phi^{-1}(B_{\lambda-1}) = \phi_j^{-1}(A_\lambda)$, in the notation of lemma 1.

We shall first construct a dynamical system on $\mathbb{K}^N$. An element of $\mathbb{K}^N$ will be written $(x_1, ... x_N)$, where each $x_k \in \mathbb{R}$ is interpreted as its fractional part and the group operation is addition (mod one). Since ϕ is an order preserving bijection we may, unambiguously, write $x^*(i_1, ... i_m)^*$ to mean x_k if $\phi(k) = (i_1, ... i_m)$. Thus, in effect, $* .. *$ is the map ϕ^{-1} followed by movement down the page by two millimetres.

We now define a dynamical system with generators T_j, $1 \leq j \leq m$. If $x \in \mathbb{K}^N$ define the k'th co-ordinate of $T_j(x)$ by;

$$(T_j(x))_k = x_k + a(\phi_j(k)) + x^*(\phi(k) + \underline{j})^* .$$

If $\phi(k) + \underline{j} \notin I$ then $x^*(\phi(k) + \underline{j})^* = 0$. We note that T_j is invertible. In fact;

$$(T_j^{-1}(x))_k = x_k - x^*(\phi(k) + \underline{j})^* + x^*(\phi(k) + 2.\underline{j})^* - \ldots$$
$$- a(\phi_j(k)) + a(\phi_j(k) + \underline{j}) - \ldots$$

We denote the binomial coefficient $p!/q!(p-q)!$ by $\binom{p}{q}$ and adopt the usual conventions $0! = 1$ and $\binom{p}{q} = 0$ if $q > p$ or $q < 0$. Recall that $\binom{p}{q} + \binom{p}{q+1} = \binom{p+1}{q+1}$. (This is the defining property of Pascal's triangle.)

<u>Lemma 2.</u> $$(T_j^{\,p}(x))_k = x_k + \sum_{q\geq 1} \binom{p}{q} x^*(\phi(k) + q.\underline{j})^* + \sum_{q\geq 0} \binom{p}{q+1} a(\phi_j(k) + q.\underline{j})$$

for all integers $p \geq 0$.

<u>Proof.</u> The lemma is certainly true if $p = 1$ as it is then just the definition of $(T_j(x))_k$.

Assume, as an induction hypothesis, it is true for some $p \geq 1$.

Then, by definition, $(T_j(T_j^{\,p}(x)))_k =$

$$x_k + \sum_{q\geq 1} \binom{p}{q} x^*(\phi(k) + q.\underline{j})^* + \sum_{q\geq 0} \binom{p}{q+1} a(\phi_j(k) + q.\underline{j})$$
$$+ (T_j^{\,p}(x))^*(\phi(k) + \underline{j})^* + a(\phi_j(k)).$$

$$= x_k + \sum_{q\geq 1} \binom{p}{q} x^*(\phi(k) + q.\underline{j})^* + \sum_{q\geq 0} \binom{p}{q+1} a(\phi_j(k) + q.\underline{j})$$
$$+ x^*(\phi(k) + \underline{j})^* + \sum_{q\geq 1} \binom{p}{q} x^*(\phi(k) + (q+1).\underline{j})^*$$
$$+ \sum_{q\geq 0} \binom{p}{q+1} a(\phi_j(k) + (q+1).\underline{j}) + a(\phi_j(k))$$

(using lemma 1.)

$$= x_k + \sum_{q\geq 1} \left(\binom{p}{q} + \binom{p}{q-1}\right) x^*(\phi(k) + q.\underline{j})^*$$

$$+ \sum_{q\geq 0} \left(\binom{p}{q+1} + \binom{p}{q}\right) a(\phi_j(k) + q.\underline{j}).$$

$$= x_k + \sum_{q\geq 1} \binom{p+1}{q} x^*(\phi(k) + q.\underline{j})^*$$

$$+ \sum_{q\geq 0} \binom{p+1}{q+1} a(\phi_j(k) + q.\underline{j}).$$

From lemma 1 we see that lemma 2 may be phrased slightly differently. The following is the form in which it is used in the proof of lemma 3.

Lemma 2. $(T_j^p(x))_k = \sum_{q\geq 0} \binom{p}{q} x^*(\phi(k) + q.\underline{j})^*$

$$+ \sum_{q\geq 1} \binom{p}{q} a(\phi(k) + q.\underline{j}) .$$

Lemma 3. For any $j_1, \ldots j_m \in \{1, \ldots m\}$ and for all integers $p_1, \ldots p_m \geqslant 0$;

$$(T_{j_1}^{p_1} \circ T_{j_2}^{p_2} \circ \ldots \circ T_{j_m}^{p_m}(x))_k =$$

$$\sum_{q_a\geqslant 0} \binom{p_1}{q_1}\binom{p_2}{q_2} \ldots \binom{p_m}{q_m} x^*(\phi(k) + q_1.\underline{j}_1 + \ldots + q_m.\underline{j}_m)^*$$

$$+ \sum \binom{p_1}{q_1}\binom{p_2}{q_2} \ldots \binom{p_m}{q_m} a(\phi(k) + q_1.\underline{j}_1 + \ldots + q_m.\underline{j}_m) ,$$

where the undefined sum is over all $q_1, \ldots q_m \geqslant 0$ such that $q_1 + \ldots + q_m \geqslant 1$

<u>Proof.</u> This result has already been proved in the special case

$p_1 = p_2 = \dots = p_m = 0$, as lemma 2.

To prove this lemma we assume there exists $1 \leq i \leq m$ such that the result is true if $p_1 = \dots = p_i = 0$ and deduce it is true if $p_1 = \dots = p_{i-1} = 0$.

The assumption is that $(T_{j_{i+1}}^{p_{i+1}} \circ T_{j_{i+2}}^{p_{i+2}} \circ \dots \circ T_{j_m}^{p_m}(x))_k =$

$$\sum_{q_a \geq 0} \binom{p_{i+1}}{q_{i+1}} \dots \binom{p_m}{q_m} x^*(\phi(k) + q_{i+1} \cdot j_{i+1} + \dots + q_m \cdot j_m)^*$$

$$+ \quad \sum \binom{p_{i+1}}{q_{i+1}} \dots \binom{p_m}{q_m} a(\phi(k) + q_{i+1} \cdot j_{i+1} + \dots + q_m \cdot j_m) ,$$

where the undefined sum is over all $q_{i+1}, \dots q_m \geq 0$ such that $q_{i+1} + \dots + q_m \geq 1$.

Using lemma 2 we see; $(T_{j_i}^{p_i} \circ T_{j_{i+1}}^{p_{i+1}} \circ \dots \circ T_{j_m}^{p_m}(x))_k =$

$$\sum_{q_i \geq 0} \binom{p_i}{q_i} \left[\sum_{q_a \geq 0} \binom{p_{i+1}}{q_{i+1}} \dots \binom{p_m}{q_m} x^*(\phi(k) + q_{i+1} \cdot j_{i+1} + \dots \right.$$

$$\dots + q_m \cdot j_m + q_i \cdot j_i)^*$$

$$+ \quad \sum \binom{p_{i+1}}{q_{i+1}} \dots \binom{p_m}{q_m} a(\phi(k) + q_{i+1} \cdot j_{i+1} + \dots$$

$$\left. \dots + q_m \cdot j_m + q_i \cdot j_i) \right]$$

$$+ \quad \sum_{q_i \geq 1} \binom{p_i}{q_i} a(\phi(k) + q_i \cdot j_i) ,$$

where the undefined sum is as above on this page.

$$= \sum_{q_a \geq 0} \binom{p_i}{q_i} \dots \binom{p_m}{q_m} x^*(\phi(k) + q_i \cdot \underline{i}_i + \dots + q_m \cdot \underline{i}_m)^*$$

$$+ \sum \binom{p_i}{q_i} \dots \binom{p_m}{q_m} a(\phi(k) + q_i \cdot \underline{i}_i + \dots + q_m \cdot \underline{i}_m) ,$$

where the undefined sum is over all $q_i, \dots q_m \geq 0$ such that $q_i + \dots + q_m \geq 1$.

One particular consequence of this lemma is that we see for any j_1 and j_2; $(T_{j_1} \circ T_{j_2}(x))_k =$

$$x^*(\phi(k) + \underline{i}_1 + \underline{i}_2)^* + x^*(\phi(k) + \underline{i}_1)^* + x^*(\phi(k) + \underline{i}_2)^*$$
$$+ x^*(\phi(k))^* + a(\phi(k) + \underline{i}_1 + \underline{i}_2) + a(\phi(k) + \underline{i}_1)$$
$$+ a(\phi(k) + \underline{i}_2)$$

$$= (T_{j_2} \circ T_{j_1}(x))_k , \quad \text{for all } 1 \leq k \leq N$$

The group action is abelian and, hence, a factor group of $\mathbb{Z}^m$ is isomorphic to the group acting. We use here the classification theorem of finitely generated abelian groups which is described in, for instance, (17). We shall use it again without specifically mentioning the fact.

In order to construct a dynamical system with the required properties we must restrict the space on which the group acts to a closed sub-group, X, of $\mathbb{K}^N$. Th action described on $\mathbb{K}^N$ is only rarely minimal. To use a result of Hoare and Parry (15) on the ergodicity of semi-groups of affine transformations X must also be a connected sub-group.

If X is any closed sub-group of $\mathbb{K}^N$ then X is isomorphic (both algebraically and topologically) to $\mathbb{K}^M \times D$, for some $0 \leq M \leq N$ and finite, discrete abelian group D. This fact follows from a consideration of the dual group $\hat{X}$ which is isomorphic to a factor group of $\mathbb{Z}^N$. Thus $\hat{X}$ is isomorphic to $\mathbb{Z}^M \times E$, for some $0 \leq M \leq N$ and finite abelian group E. By duality X is isomorphic to $\mathbb{K}^M \times \hat{E}$.

Let $\mathbb{K}^N = \mathbb{K}_1 \times \mathbb{K}_2 \times \ldots \times \mathbb{K}_N$. If $Y_k \subset \mathbb{K}_k$ for all $1 \leq k \leq N$ and $K = k : \{Y_k = \{0\}\}$ then we often write $Y_1 \times Y_2 \times \ldots \times Y_N$ omitting Y_k if $k \in K$. $[Y]$ will denote the smallest closed sub-group of $\mathbb{K}^N$ containing the set $Y \subset \mathbb{K}^N$. $[Y] = \{y \in \mathbb{K}^N : y = \lim_{i \to \infty} y_i$, where each y_i is a finite sum of integer multiples of elements of $Y\}$.

We recall some of the notation of the proof of lemma 1. I is partitioned into disjoint sets; $I = B_{n-1} \cup B_{n-2} \cup \ldots \cup B_0$, where $B_{\lambda-1} = \Phi(Q_\lambda)$. Thus;

$$\begin{aligned} B_{\lambda-1} &= \{(i_1, \ldots i_m) \in I : i_1 + \ldots + i_m = \lambda - 1\} \\ &= \{(i_1, \ldots i_m) : (i_1, \ldots i_m) + \underline{j} \in \Phi_j(Q_\lambda)\} \end{aligned}$$

Let $N(\lambda) = |B_{\lambda-1}| = |Q_\lambda|$ and $P_\lambda : \mathbb{K}^N \to \mathbb{K}^{N(\lambda)}$ be the projection map;

$$x_k \longrightarrow \begin{cases} x_k & \text{if } k \in Q_\lambda \\ 0 & \text{if } k \notin Q_\lambda \end{cases}$$

For each $1 \leq j \leq m$ we define an element of $\mathbb{K}^N$; $s_j = (a_1, \ldots a_N)$, where $a_k = a(\Phi_j(k))$. Given $x \in \mathbb{K}^N$ let $x'(j) = (x'_1, \ldots x'_N)$ be defined by;

$$x'_k = \begin{Bmatrix} x^*(\Phi(k) + j)^* & \text{if} \quad k \notin Q_n \\ 0 \quad \text{if} \quad k \in Q_n \end{Bmatrix} = (T_j(x))_k - (x_k + a(\Phi_j(k)))$$

We define $X_n = \left[\left\{P_n(s_j) : 1 \leq j \leq m\right\}\right]$.

Given $X_{\lambda+1}$ we define X_λ to equal;

$$\left[\left\{P_\lambda(s_j) : 1 \leq j \leq m\right\} \cup \bigcup_{1 \leq j \leq m} \left\{P_\lambda(x'(j)) : x \in X_{\lambda+1}\right\}\right]$$

Then let $X = X_n \times X_{n-1} \times \ldots . \times X_1$ so X is a closed sub-group of $\mathbb{K}^N$.

<u>Lemma 4.</u> (1). If $x \in X$ then for all $1 \leq j \leq m$; $T_j(x) \in X$ and also $T_j^{-1}(x) \in X$.

(2). Suppose there exists some $k_0 \in Q_\lambda$ and $1 \leq j \leq m$ such that $a(\Phi_j(k_0))$ is irrational. Then the projection of X on the co-ordinate k_0 is an onto map.

(3). Suppose there exists some $k_0 \in Q_\lambda$, for some $\lambda \geq 2$, and $1 \leq j \leq m$ such that $\Phi(k_0) - j \in I$ and the projection of X_λ on the co-ordinate k_0 is onto. Then the projection of $X_{\lambda-1}$ on the co-ordinate $\Phi(k_0) - j$ is also an onto map.

<u>Proof.</u> (1). This is clear from the definitions of X and T_j.

(2). X_λ contains the discrete sub-group generated by the element $P_\lambda(s_j)$.
Since $a(\Phi_j(k_0))$ is irrational the projection of this sub-group on k_0 is dense, and the result follows.

(3). $X_{\lambda-1}$ contains the subset; $\left\{P_{\lambda-1}(x'(j)) : x \in X_\lambda\right\}$.

Since the value of $x^*(\Phi(k_0))^*$ can be any real number between zero and one the projection of this subset on the co-ordinate $\Phi(k_0) - j$ is onto.

<u>Corollary.</u> $X_1 = \mathbb{K}_N$.

<u>Proof.</u> We have assumed that at least one of the numbers;

$$\left\{ a(i_1, \ldots i_m) : (i_1, \ldots i_m) \text{ is allowed} \right\}$$

is irrational.

Use part (2) of the lemma and then part (3) sufficiently many times.

From now on we shall consider the group action with generators $T_1, \ldots T_m$ acting on the closed sub-group X. This dynamical system may be denoted (T, X). We have already remarked that the group acting is a factor of $\mathbb{Z}^m$. In certain cases we have actually constructed an action of $\mathbb{Z}^m$ itself. For instance we have the following result.

<u>Lemma 5.</u> Suppose that the set;

$$\left\{ a(i_1, \ldots i_m) : (i_1, \ldots i_m) \text{ is allowed} \right\}$$

has the following special form. There exists an integer n', where $1 \leq n' \leq n$, and irrationals $\alpha_1, \ldots \alpha_m$ such that

(1) For each j; $a(n'.j) = \alpha_j$

(2) If not defined by (1) then; $a(i_1, \ldots i_m) = 0$.

Then (T, X) is an effective action of the group $\mathbb{Z}^m$ on X. Furthermore

X is a closed connected group isomorphic to $\mathbb{K}^M$, where $M = (n' - 1)m + 1$.

Proof. We first show that X_λ is isomorphic to $\mathbb{K}^m$ if $n' \geqslant \lambda \geqslant 2$.

Clearly $X_\lambda = \{0\}$ if $\lambda > n'$ and the corollary to lemma 4 states $X_1 = \mathbb{K}$, so this suffices to prove X is isomorphic to $\mathbb{K}^M$, and hence connected.

$X_{n'} = \{P_{n'}(s_j) : 1 \leqslant j \leqslant m\}$.

If $j' \neq j$ then; $n'.\underline{j}' \notin I_j$ so $a(\phi_j(k)) = 0$ unless $\phi_j(k) = n'.\underline{j}$.

Thus $X_{n'} = \mathbb{K}^*((n'-1).\underline{m})^* \times \ldots\ldots \times \mathbb{K}^*((n'-1).\underline{1})^*$,

since, by lemma 1, $\phi_j(k) = n'.\underline{j}$ if and only if $\phi(k) = (n'-1).\underline{j}$.

$$X_{n'-1} = \left[\bigcup_{1 \leqslant j \leqslant m} \{P_{n'-1}(x'(j)) : x \in X_{n'}\} \right].$$

Thus if $n' \geqslant 3$ we deduce;

$$X_{n'-1} = \mathbb{K}^*((n'-2).\underline{m})^* \times \ldots\ldots \times \mathbb{K}^*((n'-2).\underline{1})^*.$$

Clearly a similar expression describes X_λ if $2 \leqslant \lambda \leqslant n'$.

From the definition of T_j we note that if $k \in Q_{n'}$ then $(T_j(x))_k = x_k$ unless $\phi(k) = (n'-1).\underline{j}$ in which case $(T_j(x))_k = x_k + \alpha_j$.

$$(T_1^{p_1} \circ \ldots\ldots \circ T_m^{p_m}(x))^*((n'-1).\underline{j})^* = x^*((n'-1).\underline{j})^* + p_j \alpha_j,$$

for all $p_1, \ldots p_m$.

Thus $(T_1^{p_1} \circ \ldots \circ T_m^{p_m}(x)) = x$, for all $x \in X$, implies $p_1 = p_2 = \ldots = p_m = 0$.

4. Properties of Connected (T , X).

Throughout this sub-section we shall <u>assume X is connected</u>. This is certainly possible, as is shown by lemma 3.5. As already remarked the reason for assuming connectedness is that we wish to use a result of Hoare and Parry (15) on the ergodicity of semi-groups of affine transformations.

<u>Notation</u> Y is a compact, connected abelian group with group operation; $+$.

$\mathcal{T}$ denotes an abelian semi-group of affine transformations of Y; $T(y) = s + S(y)$, where $s \in Y$ and $S: Y \to Y$ is a group automorphism.

$\Gamma = \hat{Y}$ and $\hat{S}: \Gamma \to \Gamma$ is defined by $\hat{S}(\gamma) = \gamma \circ S$.

$$\mathcal{S} = \{ S : \cancel{s + S \in \mathcal{T}} \ \exists\, s \in Y \text{ such that } s + S \in \mathcal{T} \}$$

$$Y' = \{ s : \cancel{s + S \in \mathcal{T}} \ \exists\, S : Y \to Y \text{ such that } s + S \in \mathcal{T} \}$$

If $T = s + S$ then denote T^p by $s^p + S^p$.

Any affine transformation of a compact group certainly preserves Haar measure, m. We normalise so $m(Y) = 1$.

<u>Theorem.</u> (Hoare and Parry) $\mathcal{T}$ is ergodic with respect to Haar measure if and only if;

(1) For every $\gamma \in \Gamma$; $\{ \hat{S}(\gamma) : S \in \mathcal{S} \}$ is either an infinite set or has cardinality one.

and (2) $\left[Y' \cup \{ S(y) - y : y \in Y, S \in \mathcal{S} \} \right] = Y.$

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and (2) $[Y' \cup \{ S(y) - y : y \in Y, S \in \mathcal{S} \}] = Y.$

<u>Lemma 1.</u> If $\mathcal{T}$ is finitely generated by the transformations $T_j = s_j + S_j$, $1 \leq j \leq m$, then (2) is equivalent to;

$$(2') \quad \left[\{s_j : 1 \leq j \leq m\} \cup \{S_j(y) - y : y \in Y,\ 1 \leq j \leq m\}\right] = Y.$$

<u>Proof.</u> Let Y_0 be the group generated by;

$$\{s_j : 1 \leq j \leq m\} \cup \{S_j(y) - y \ : y \in Y,\ 1 \leq j \leq m\} .$$

(Y_0 is not in general closed.)

We show that if s', $s'' \in Y_0$ and $S'(y) - y$, $S''(y) - y \in Y_0$ for all $y \in Y$ and $(s' + S')\circ(s'' + S'') = s + S$ then $s \in Y_0$ and $S(y) - y \in Y_0$ for all $y \in Y$.

Hence $Y_0 \supset Y' \cup \{S(y) - y : y \in Y,\ S \in \mathcal{S}\}$ and the lemma is proved.

$$(s' + S')\circ(s'' + S'') = s' + S'(s'') + S'\circ S''.$$

Thus $s = s' + S'(s'') = s' + S'(s'') - s'' + s'' \in Y_0$.

Also $S'\circ S''(y) - y = S'(S''(y)) - S''(y) + S''(y) - y \in Y_0$ for all $y \in Y$.

<u>Definition.</u> A group $\mathcal{T}$ of transformations is <u>totally ergodic</u> with respect to a measure, m, if every syndetic sub-group of $\mathcal{T}$ is ergodic with respect to m.

The ergodicity of a finitely generated group of transformations is clearly equivalent to the ergodicity of the semi-group with the same generators. In other words we need only consider positive powers of the transformations T_j constructed in the last sub-section to prove ergodicity. First, however, we note a more general result.

Proposition 7. Let $\mathcal{T}$ be a finitely generated group of affine transformations of the compact connected group Y. Then $\mathcal{T}$ is ergodic, with respect to Haar measure, if and only if it is totally ergodic.

Proof. We have to show ergodicity implies total ergodicity, as the converse is immediate.

Let $\mathcal{T}$ have generators $T_j = s_j + S_j$, $1 \leq j \leq m$.

Any syndetic sub-group of $\mathcal{T}$ contains another of the form $\{ T^p : T \in \mathcal{T} \}$ for some $p \geq 1$. (see (17).)

It therefore suffices to show conditions (1) and (2) of the theorem of Hoare and Parry imply;

(1,p) For every $\gamma \in \Gamma$; $\{ \hat{S}^p(\gamma) : S \in \mathcal{S} \}$ is either an infinite set or has cardinality one.

(2',p) $[\{ s_j^p : 1 \leq j \leq m\} \cup \{ S_j^p(y) - y : y \in Y, 1 \leq j \leq m\}] = Y$.

for every $p \in N^+$.

(1,p) is trivial so we prove (2',p).

Let $\gamma \in \Gamma$ be any character such that $\gamma(s_j^p) = 1$, $1 \leq j \leq m$, and $\gamma(S_j^p(y) - y) = 1$, for all $y \in Y$ and $1 \leq j \leq m$.

We show that $\gamma \equiv 1$.

$\gamma(S_j^p(y)) = \gamma(y)$ for all $y \in Y$ and $1 \leq j \leq m$.

So $\gamma \circ S = \gamma$ and also $\gamma^p \circ S = \gamma^p$, for all $S \in \mathcal{S}$, by (1).

$s_j^p = s_j + S_j(s_j) + \ldots + S_j^{p-1}(s_j)$.

$\gamma(s_j^p) = \gamma(s_j + S_j(s_j) + \ldots + S_j^{p-1}(s_j)) = \gamma^p(s_j)$.

Thus $\gamma^p(y) = 1$ if $y \in [\{ s_j : 1 \leq j \leq m\} \cup \{ S_j(y) - y : y \in Y, 1 \leq j \leq m\}]$ so $\gamma^p \equiv 1$, by (2').

Since Y is connected Γ is torsion free so $\gamma \equiv 1$.

The transformations T_j, $1 \leq j \leq m$, defined in sub-section 3 are affine transformations of X (and of $\mathbb{K}^N$ for that matter). Writing $T_j = s_j + S_j$ we see $(S_j(x))_k = x_k + x^*(\phi(k) + \underline{j})^*$ and $(s_j)_k = a(\phi_j(k))$ in this case. These are the meanings attached to these symbols from now on. Regarding, for the moment, S_j as an automorphism of $\mathbb{K}^N$; $\hat{S}_j$ is an automorphism of $\mathbb{Z}^N$. In fact;

$$\begin{aligned}(\hat{S}_j(z))(x) &= z(S_j(x)) \\ &= \exp\left(\sum_{k=1}^{N} z_k(x_k + x^*(\phi(k) + \underline{j})^*)\right) \\ &= \exp\left(\sum_{k=1}^{N} (z_k + z^*(\phi(k) - \underline{j})^*)x_k\right)\end{aligned}$$

Here we are denoting an element of $\mathbb{Z}^N$ by $(z_1, \ldots z_N)$ and z_k by $z^*(\phi(k))^*$ in a manner similar to the notation for elements of $\mathbb{K}^N$. Thus $(\hat{S}_j(z))_k = z_k + z^*(\phi(k) - \underline{j})^*$. The proof of the next lemma is so similar to that of lemmas 3.2 and 3.3 that it is not worth reproducing.

<u>Lemma 2.</u> $(\hat{S}_1^{p_1} \ldots \hat{S}_m^{p_m}(z))_k =$

$$\sum_{q_a \geq 0} \binom{q_1}{p_1} \ldots \binom{q_m}{p_m} z^*(\phi(k) - q_1.\underline{1} - \ldots - q_m.\underline{m})^*$$

Regarding S_j as an automorphism of X, $\hat{S}_j$ is an automorphism of the factor group $\hat{X}$ of $\mathbb{Z}^N$ defined in exactly the way already described but on coset representatives. Our assumption that X is connected (in fact isomorphic to $\mathbb{K}^M$ for some $1 \leq M \leq N$) is equivalent to assuming $\hat{X}$ is torsion free (in fact isomorphic to $\mathbb{Z}^M$)

Lemma 3. For each j, $1 \le j \le m$, and each $z \in X$ the cardinality of the set $\{\hat{S}_j^{\,p}(z) : p \in \mathbb{N}\}$ is either infinite or one

Proof. $X = \hat{X}_n \times \ldots \times \hat{X}_1$, where each $\hat{X}_\lambda$ is torsion free.

We write $z \in \hat{X}$ as an element of $\mathbb{Z}^N$ though in fact it is the coset containing this element.

Let $P_\lambda : \hat{X} \to \hat{X}_\lambda$ denote projection onto $\hat{X}_\lambda$.

Then $z = (P_n(z), \ldots P_1(z))$.

If $\lambda \geq 2$ we define an element of $\hat{X}_\lambda$; $z'(\lambda, j) = (\ldots z'_k, \ldots)$, $k \in Q_\lambda$, by;

$$z'_k = \begin{cases} z^*(\phi(k) - j)^* & \text{if } (\phi(k) - j) \in I \\ 0 & \text{if not} \end{cases}$$

Then, by lemma 2, we have;

$$\hat{S}_j(z) = (P_n(z) + z'(n,j), \ldots P_2(z) + z'(2,j), P_1(z))$$

Either there exists $2 \le \lambda \le n$ such that $z'(\lambda, j)$ is an element of a non-zero coset in $\hat{X}_\lambda$ or z is a fixed point of $\hat{S}_j$.

In the first case let λ_0 be the least such λ.

By lemma 2 we have;

$$P_{\lambda_0}(\hat{S}_j^{\,p}(z)) = P_{\lambda_0}(z) + p.z'(\lambda_0, j)$$

Thus $\{\hat{S}_j^{\,p}(z) : p \in \mathbb{N}\}$ certainly has infinitely many elements.

Theorem 8. (T , X) is totally ergodic, with respect to Haar measure.

Proof. By proposition 7 it suffices to prove ergodicity.

Lemma 3 implies condition (1) of the theorem of Hoare and Parry is satisfied.

The construction of X is precisely so condition (2) is satisfied.

Theorem 9. (T , X) has quasi-discrete spectrum. In fact all the elements of $\mathbb{K} + (\hat{X}_n \times \ldots \times \hat{X}_{n-n'})$ are quasi eigenfunctions of order $n' + 1$ and the group of quasi eigenfunctions is $\mathbb{K} + \hat{X}$.

Proof. We continue to regard $\mathbb{K}$ as the interval $[0,1)$ with group operation addition and the topology of the set of complex numbers of absolute value one.

This forces the use of additive notation when multiplicative is more natural.

Let $z \in \hat{X}_n \times \ldots \times \hat{X}_{n-n'}$ and $x \in \mathbb{K}$.

In the notation of lemma 3 we have, for all $1 \leq j \leq m$;

$$\hat{S}_j(x + z) - (x + z) = (z'(n,j), \ldots z'(2,j), 0) \in \hat{X}_n \times \ldots \times \hat{X}_{n-n'+1}$$

provided $n' \neq 0$.

If $n' = 0$ then; $\hat{S}_j(x + z) - (x + z) = 0$.

Thus if $n' \neq 0$ we have;

$$(x + z)_\circ T_j - (x + z) \in \mathbb{K} + (\hat{X}_n \times \ldots \times \hat{X}_{n-n'+1})$$

and if $n' = 0$; $(x + z)_\circ T_j - (x + z) \in \mathbb{K}$.

From lemma 1.1 we see each element of

$\mathbb{K} + (\hat{X}_n \times \dots \times \hat{X}_{n-n'})$ is indeed a quasi eigenfunction

of order $n' + 1$.

Hence, by the Stone-Wierstrass theorem, (T, X) has

quasi-discrete spectrum.

From theorem 5 we see (T, X) satisfies hypothesis (A)

of theorem 2 and hence the conclusion of that theorem.

No L^2 function can be orthogonal to $\hat{X}$ so we have identified

all the quasi eigenfunctions.

We are now able to state and prove the theorem about the existence of $\mathbb{Z}^m$ - actions promised in subsection 2. It can be easily proved directly.

<u>Theorem 10.</u> Given any $m \in \mathbb{N}^+$ and $n \in \mathbb{N}^+$ there exists an effective uniquely ergodic $\mathbb{Z}^m$ - action with quasi-discrete spectrum. Furthermore this dynamical system has quasi eigenfunctions which are of minimum order n, ~~but no lesser order.~~

<u>Proof.</u> Consider the dynamical system described in lemma 3.5,

with $n = n'$.

It has quasi-discrete spectrum (theorem 9).

It satisfies hypothesis (A) of theorem 2 (theorem 8).

It is uniquely ergodic (theorem 2).

It only remains to show there are quasi-eigenfunctions of

order n but not $n - 1$.

Observe, from the proof of lemma 3.5, that $\hat{X} =$

$\hat{X}_n \times \dots \times \hat{X}_1$, where if $\lambda \neq 1$;

$$\hat{X}_\lambda = \mathbb{Z}^*((\lambda-1).\underline{m})^* \times \dots \times \mathbb{Z}^*((\lambda-1).\underline{1})^*$$

$\hat{X}_1 = \mathbb{Z}$, so it may be described by the above expression also provided it is remembered that $0.\underline{j} = (0, \dots 0)$ for all j.

Let z be a non zero element of $\hat{X}_\lambda$ so there exists some j such that $z*((\lambda - 1).\underline{j})* \not\leq 0$.

Then, by lemma 2, $\hat{S}_j(z) - z$ is a non zero element of $\hat{X}_{\lambda+1}$.

In fact $\hat{S}_j(z) - z \in \mathbb{Z}*(\lambda.\underline{j})*$.

Thus we conclude any non zero element of $\hat{X}_1$ has only orders greater than or equal to n.

We now turn to the demonstration, promised in subsection 1, that hypothesis (2) of theorem 5 cannot be deduced from (1) and (3) in general. This is contrary to the hopes expressed in (12) and (20). We continue all the notation of subsection 1.

Let $n = n' = 3$, $m = 2$ and consider a dynamical system, (T, X) constructed as in lemma 3.5. We note a number of facts about this dynamical system.

From the proof of lemma 3.5 observe that;

$$X_3 = \mathbb{K}_1 \times \mathbb{K}_3, \quad X_2 = \mathbb{K}_4 \times \mathbb{K}_5, \quad X = \mathbb{K}^5 \qquad (*)$$

The proof of theorem 10 demonstrates;

$$G_1 = \mathbb{K} + \hat{X}_3 = \mathbb{K} + \mathbb{Z}_1 \times \mathbb{Z}_3 \qquad (**)$$

The proof of lemma 3.5 also shows;

$$\begin{aligned} (T_1^{p_1} . T_2^{p_2} x)_1 &= x_1 + p_2 . \alpha_2 \\ (T_1^{p_1} . T_2^{p_2} x)_3 &= x_3 + p_1 . \alpha_1 \end{aligned} \qquad (***)$$

for all $p_1, p_2 \in \mathbb{Z}$.

Now also consider another dynamical system, (T', X'), constructed from the set; $\{a(i_1,i_2) : (i_1,i_2) \text{ is allowed}\}$, where $n = 3$. We suppose that $a(3,0) = \alpha_1$, $a(0,3) = \alpha_2$ and $a(i_1,i_2) = 0$ if $i_1 + i_2 = 3$ and $i_1, i_2 > 0$. However in this case we do not restrict $a(i_1,i_2)$ if $i_1 + i_2 \leq 2$. For the same reasons as above statements (*), (**) and (***) are also true about (T', X').

$S_j = S'_j$, $j = 1,2$, and $\hat{X} = \Gamma = \Gamma' = \hat{X}'$ so $\sigma_t = \sigma'_t$ and the isomorphism $V = \tau \circ \tau'^{-1}$ satisfies condition (1) of theorem 5. Recall, from page 3-1-10, that if $g \in G_1 \cap \Gamma$ then $\tau(g)$ is the map $t \longmapsto g(s_t)$. Thus (**) and (***) show that V satisfies (3).

In this case (2) is equivalent to;

(2') there exists $x \in X'$ such that $s_t - s'_t = S'_t(x) - x$ for all $t \in \mathbb{Z}^2$,

since we are now using additive (mod one) notation. In particular we can show (2) does not hold by showing there is no solution to (2') for all $t \in \mathbb{N}^2$. In other words, from lemma 3.3, we claim there is no solution, $x = (x_1, x_3, x_4, x_5, x_6)$, to the equations;

$$\left.\begin{array}{l} 0 \\ 0 \\ p_2 \cdot x_1 \\ p_1 \cdot x_3 \\ p_2 \cdot x_4 + p_1 \cdot x_5 + \binom{p_2}{2} \cdot x_1 + \binom{p_1}{2} \cdot x_3 \end{array}\right\} =$$

$p_2.(\alpha_2 - \alpha_2)$

$p_1.(\alpha_1 - \alpha_1)$

$\binom{p_2}{2}(\alpha_2 - \alpha_2) - p_2.a(0,2) - p_1.a(1,1)$

$\binom{p_1}{2}(\alpha_1 - \alpha_1) - p_1.a(2,0) - p_2.a(1,1)$

$$\binom{p_2}{3}(\alpha_2 - \alpha_2) + \binom{p_1}{3}(\alpha_1 - \alpha_1) - \binom{p_2}{2}.a(0,2) - \binom{p_1}{2}.a(2,0)$$
$$- p_1.p_2.a(1,1) - p_2.a(0,2) - p_1.a(2,0)$$

which is certainly the case for a suitable choice of elements of $\left\{a(i_1,i_2) : 1 \leq i_1 + i_2 \leq 2\right\}$.

5. Weyl's Theorem.

In this subsection we use the previous work to give a new proof of the classical theorem of H. Weyl which states that a real polynomial of finitely many integer variables having at least one irrational coefficient (other than the constant term) defines a sequence which is uniformly distributed (mod one) (28). Historically this can be regarded as the first ergodic theorem to be discovered. In the simplest case, a polynomial of one variable of degree one, it amounts, in ergodic theoretic terms, to saying that an irrational rotation on the circle is uniquely ergodic. In (11) Hahn was able to give a proof of Weyl's theorem for polynomials of one variable using ergodic theoretic facts, thus incorporating an original inspiration of ergodic theory within its present domain. We complete the process by giving a proof which is valid for any number of variables.

Definitions. A sequence $\alpha : \mathbb{N} \to \mathbb{R}$ is uniformly distributed (mod one) if for every interval $I \subset [0,1)$ the partial sums;

$$f_I(N) = 1/N \sum_{p=0}^{N-1} \chi_I(\alpha'(p)) \longrightarrow \text{length}(I), \quad \text{as } N \longrightarrow \infty.$$

Here $\alpha'(p)$ is the fractional part of $\alpha(p)$ and χ_I is the function which is one on I and zero elsewhere.

If $\alpha : \mathbb{N}^m \longrightarrow \mathbb{R}$ let;

$$f_I(N_1, \dots N_m) = 1/N_1 \dots N_m \left(\sum_{p_1=0}^{N_1-1} \dots \sum_{p_m=0}^{N_m-1} \chi_I(\alpha'(p_1, \dots p_m)) \right).$$

α is uniformly distributed (mod one) if for every interval $I \subset [0,1)$ and every bijection $\theta : \mathbb{N} \to \mathbb{N}^m$;

$$f_I(\theta(N)) \longrightarrow \text{length}(I), \quad \text{as } N \longrightarrow \infty.$$

Equivalently one can demand this is true for some such bijection θ.

The next lemma shows how easy it is for X to be connected.

Lemma 1. If the sub-group of $\mathbb{R}$ generated by the set of numbers $\{a(i_1, \dots i_m) : (i_1, \dots i_m) \text{ is allowed}\}$ contains no rational elements other than integers then X is connected.

Proof. Refer to the definition of X in subsection 3.

Suppose for some λ, $n-1 \geq \lambda \geq 1$, $X_{\lambda+1}$ is connected but X_λ is not.

Then X_λ is isomorphic to $K^{M(\lambda)} \times D$, where D is a finite abelian group.

Hence X_λ contains an element x which possesses finite, non zero, order such that $\{x\}$ is an open set.

$x = \lim_{i \to \infty} y_i$, where each y_i is a sum of integer multiples of elements of the set defining X_λ.

For large enough i $y_i = x$ and x is a sum of integer multiples of elements of the set $\{P_\lambda(s_j) : 1 \leq j \leq m,\}$

This contradicts the hypothesis of the lemma and we conclude X_λ is also connected.

If we define $X_{n+1} = \{0\}$ then the same argument shows that X_n is connected, completing the proof of the lemma.

We consider a real polynomial of finitely many integer variables, f. Let m be the number of variables and n the degree of f. Then we can write;

$$f(q_1, \dots q_m) = \sum b(i_1, \dots i_m)\, q_1^{i_1} \dots q_m^{i_m} + b(0),$$

where the sum is over all allowed $(i_1, \dots i_m)$. We assume that at least one of the set of real numbers $\{b(i_1, \dots i_m) : (i_1, \dots i_m)$

is allowed$\}$ is irrational. Let B be the sub-group of $\mathbb{R}$ generated by $\{b(i_1, \dots i_m) : (i_1, \dots i_m)$ is allowed$\}$. B is a finitely generated abelian group and certainly contains no elements of finite order. Thus there exist rationally independent real numbers; $\beta_1, \dots \beta_r$ such that $B = \beta_1\mathbb{Z} + \dots + \beta_r\mathbb{Z}$. At most one of $\beta_1, \dots \beta_r$ is rational and if P is the denominator of that rational (if it exists) then $P.B = \{P.b : b \in B\}$, the sub-group of $\mathbb{R}$ generated by the set;

$$\{P.b(i_1, \dots i_m) : (i_1, \dots i_m) \text{ is allowed}\}$$

contains no rational elements apart from integers.

Any integer q can be expressed uniquely as; $q = p.P + h$, where $0 \leq h \leq P - 1$ and $p, h \in \mathbb{Z}$. For any choice of $h_1, \dots h_m$ such that $0 \leq h_j \leq P - 1$, $1 \leq j \leq m$, let;

$$f_{(h_1, \dots h_m)}(p_1, \dots p_m) = f(p_1.P + h_1, \dots p_m.P + h_m)$$

define a polynomial of m integer variables; $p_1, \dots p_m$, and degree n. Notice that if;

$$f_{(h_1, \dots h_m)}(p_1, \dots p_m) = \sum c(i_1, \dots i_m)\, p_1^{i_1} \dots p_m^{i_m} + c(0),$$

summing over allowed $(i_1, \dots i_m)$, then the set;

$$\{c(i_1, \dots i_m) : (i_1, \dots i_m) \text{ is allowed}\}$$

is a subset of $P.B$ containing at least one irrational. We now quote the version of the topological ergodic theorem that we use in the next lemma.

Theorem. (Krylov and Bogolioubov) Let T_j, $1 \leq j \leq m$, be commuting homeomorphisms of a compact metric space X. Then there exists at least one probability measure on the Borel subsets of X which is invarient with respect to each T_j. If $g \in \mathcal{C}(X)$ then let;

$$f_g(N_1, \dots N_m) = 1/N_1 \dots N_m \sum_{p_1=0}^{N_1-1} \dots \sum_{p_m=0}^{N_m-1} g(T_1^{p_1} \dots T_m^{p_m}(x)).$$

If there exists a unique such probability, m, then for all $g \in \mathcal{C}(X)$ and every bijection $\Theta : \mathbb{N} \to \mathbb{N}^m$;

$$f_g(\Theta(N_0)) \longrightarrow \int_X g \, dm \qquad \text{as } N_0 \to \infty .$$

Proof. The existence of an invariant measure follows from the Markov-Kakutani theorem (4), or the techniques of (18). The remainder of the theorem may be proved in exactly the same way as in the case $m = 1$ in (18).

Lemma 2. For each choice of $h_1, \dots h_m$ the polynomial $f_{(h_1, \dots h_m)} : \mathbb{N}^m \to \mathbb{R}$ is uniformly distributed (mod one).

Proof. We can rewrite the polynomial as follows;

$$f_{(h_1, \dots h_m)}(p_1, \dots p_m) = \sum a(i_1, \dots i_m) \binom{p_1}{i_1}\binom{p_2}{i_2} \dots \binom{p_m}{i_m} + a(0),$$

summing over allowed $(i_1, \dots i_m)$.

The set of numbers thus defined;

$$\left\{ a(i_1, \dots i_m) : (i_1, \dots i_m) \text{ is allowed} \right\}$$

is a subset of P.B and contains at least one irrational.

Construct the dynamical system (T , X) defined by this set of real numbers, as in subsection 3.

X is connected (Lemma 1).

(T , X) has quasi-discrete spectrum (Theorem 9).

(T , X) is totally ergodic with respect to Haar measure (Theorem 8) and therefor satisfies hypothesis (A) of theorem 2.

(T , X) is uniquely ergodic (Theorem 2).

Let $x = (0, \dots 0, x_N) \in X$, where $x_N = a(0)$ (mod one).

From lemma 3.3 we see that for all $p_1, \dots p_m \in \mathbb{N}$;

$$(T_1^{p_1} \circ \dots \circ T_m^{p_m}(x))_N = x_N +$$

$$\sum \binom{p_1}{q_1} \dots \binom{p_m}{q_m} a(\phi(N) + q_1.\underline{1} + \dots + q_m.\underline{m}),$$

summing over all integers $q_1, \dots q_m \geqslant 0$ such that $q_1 + \dots + q_m \geqslant 1$.

$$= x_N + \sum \binom{p_1}{q_1} \dots \binom{p_m}{q_m} a((0, \dots 0) + q_1.\underline{1} \dots + q_m.\underline{m}),$$

summing as above.

$$= x_N + \sum \binom{p_1}{i_1} \dots \binom{p_m}{i_m} a(i_1, \dots i_m) ,$$ summing over allowed $(i_1, \dots i_m)$.

$$= f_{(h_1, \dots h_m)}(p_1, \dots p_m) \qquad \text{(mod one)}.$$

The remainder of the argument is exactly as in (11).

For any $g \in \mathcal{C}(\mathbb{K})$ we can consider the element of $\mathcal{C}(X)$ defined by $x \to g(x_N)$.

Recall the corollary to lemma 3.4.

From the topological ergodic theorem we see that for any bijection $\Theta : \mathbb{N} \to \mathbb{N}^m$;

$$f_g(\Theta(N_0)) \longrightarrow \int_{\mathbb{K}} g \, dm \quad \text{where } m \text{ is Haar (Lebesgue) measure.}$$

Since, if $I \subset \mathbb{K}$ is any interval, χ_I can be approximated (in the sense of $L^1(m)$) from above and below by elements of $\mathcal{C}(\mathbb{K})$ we conclude that;

$$f_I(\Theta(N_0)) \longrightarrow \int_{\mathbb{K}} \chi_I \, dm \quad = \quad \text{length}(I).$$

That is to say $f_{(h_1, \ldots h_m)}$ is uniformly distributed (mod one).

<u>Theorem (</u>H. Weyl) f is uniformly distributed (mod one).

<u>Proof.</u> For a given bijection $\Theta : \mathbb{N} \to \mathbb{N}^m$ we must consider;

$$f_I(\Theta(N_0)) \quad = \quad 1/N_1 \ldots N_m \sum_{q_1=0}^{N_1-1} \cdots \sum_{q_m=0}^{N_m-1} \chi_I(\ f'(q_1, \ldots q_m)\ ),$$

where $\Theta(N_0) \;=\; (N_1, \ldots N_m)$

There exist non-negative integers η_j and π_j, $1 \leq j \leq m$, such that $0 \leq \pi_j \leq P - 1$ and $N_j = \eta_j.P + \pi_j$ so;

$$f_I(\Theta(N_0)) \;=\; 1/N_1 \ldots N_m \sum \sum_{p_1=0}^{\eta_1-1} \cdots \sum_{p_m=0}^{\eta_m-1} \chi_I(f'(p_1.P + h_1 \ldots + p_m.P + h_m))$$

plus some other terms whose sum is arbitarily small if $N_1, \ldots N_m$ are all arbitarily large,

where the sum is over all $(h_1, \ldots h_m)$.

Furthermore $(N_1 \ldots N_m) / (P^m . \eta_1 \cdots \eta_m) \;=$

$$(\eta_1.P + \pi_1) \ldots (\eta_m.P + \pi_m) / (P^m . \eta_1 \cdots \eta_m) \xrightarrow[N_0 \to \infty]{} 1.$$

Thus $\lim_{N_0 \to \infty} f_I(\Theta(N_0)) =$

$$\lim_{N_0 \to \infty} 1/P^m \sum 1/\eta_1 \cdots \eta_m \sum_{p_1=0}^{\eta_1 - 1} \cdots \sum_{p_m=0}^{\eta_m - 1} \chi_I(f'(h_1, \ldots h_m)(p_1, \ldots p_m)),$$

summing over all $(h_1, \ldots h_m)$.

$=$ length(I), by lemma 2.

NOTATION

We list here the (mostly standard) symbols which are not defined in the text.

$\mathbb{R}$ The real numbers, a group under the operation $+$.

$\mathbb{N}$ The natural numbers; $\{0, 1, 2, \ldots\ldots\}$.

$\mathbb{N}^+$ The positive natural numbers, $\{1, 2, 3, \ldots\ldots\}$.

$\mathbb{C}$ The complex numbers.

$\mathbb{K}$ $\{z \in \mathbb{C} : |z| = 1\}$, a multiplicative group.
<u>or</u>; $\{r \in \mathbb{R} : 0 \leq r < 1\}$ an additive group.

$\hat{Y}$ The character group of a group Y.

$\mathcal{P}(A)$ The power set of a set A, i.e. the set of all subsets of A.

$\prod_{i \in I} B$ The direct product of all elements of $\{B_i : i \in I\}$, where each $B_i = B$, together with the product measurable structure.

$\mathcal{C}(X)$ The group of all continuous complex valued functions on topological space X.

χ_A The characteristic function of the set A. That is the function which takes the value one on A and is zero elsewhere.

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